Extremely Deep Proofs

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Joint work with Toniann Pitassi and Robert Robere

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  → Our work based on [Razborov16]

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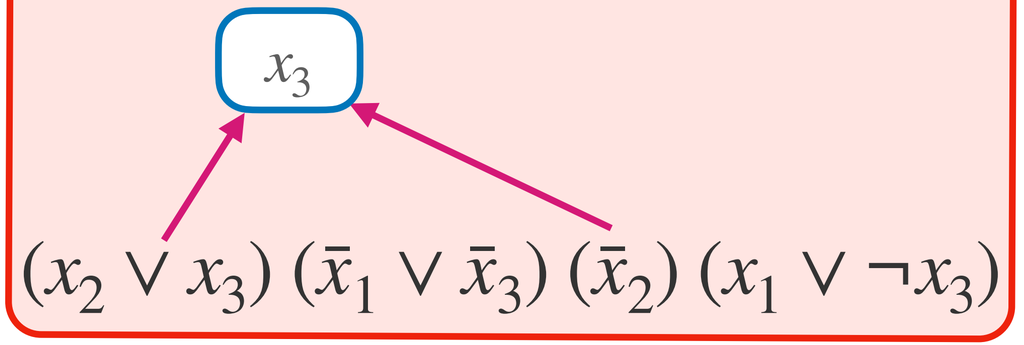

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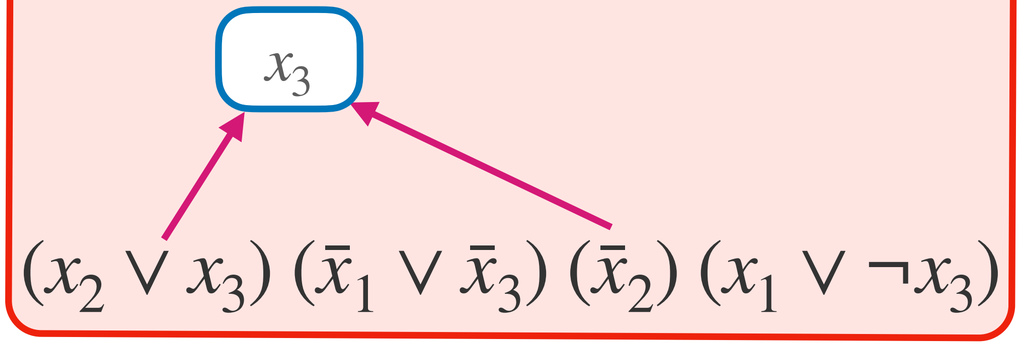

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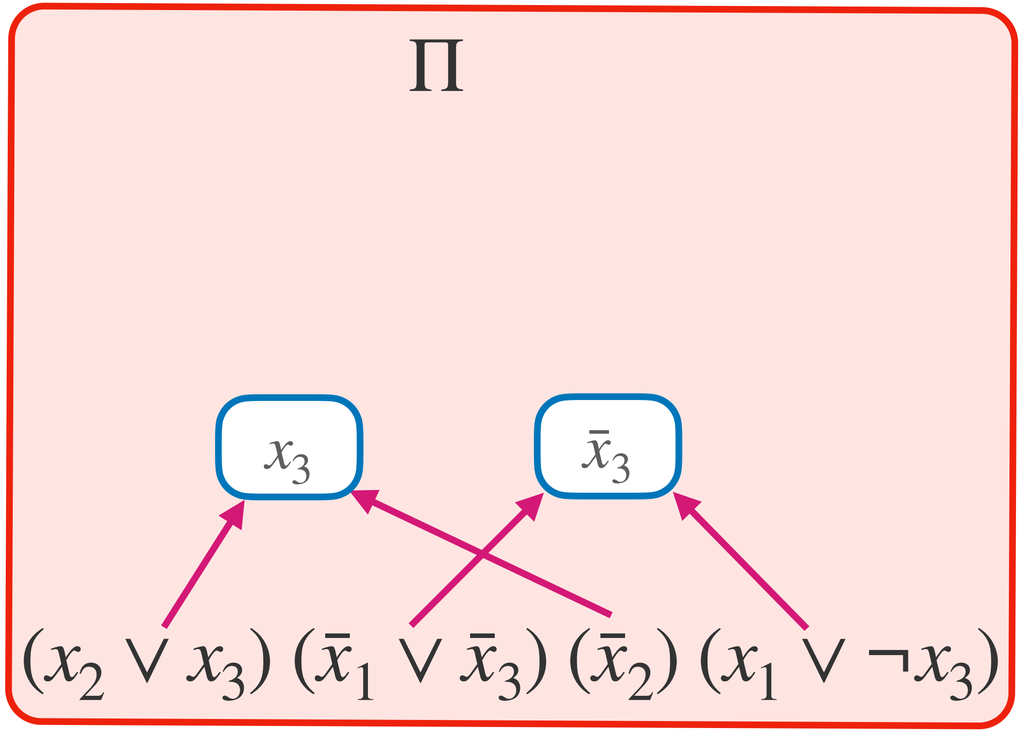

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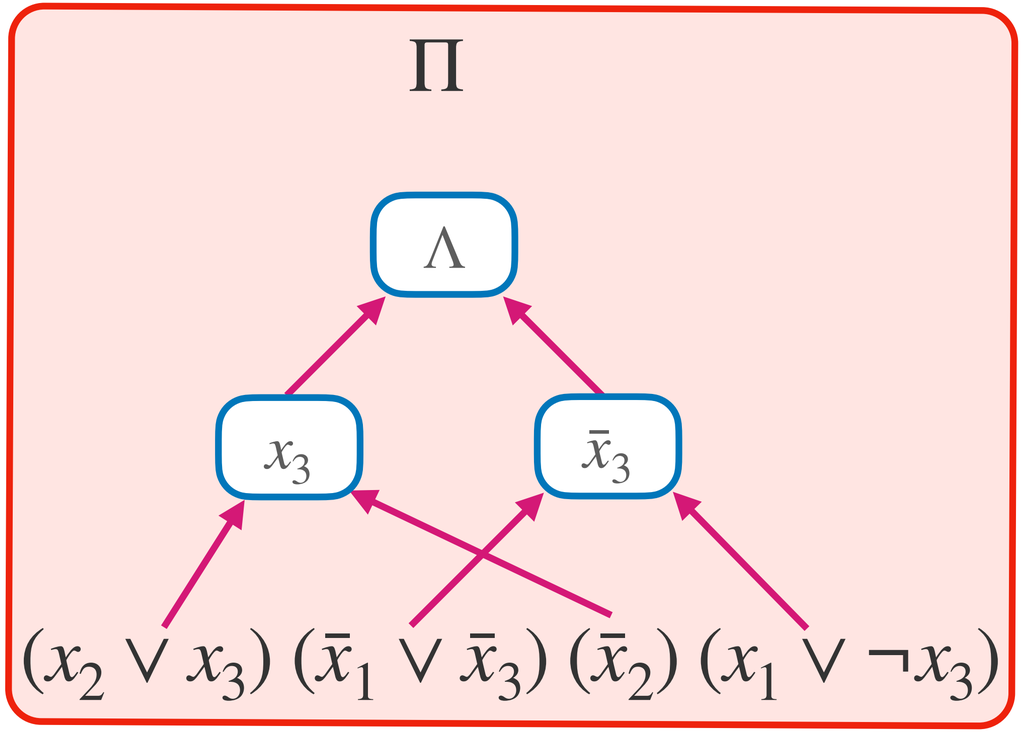

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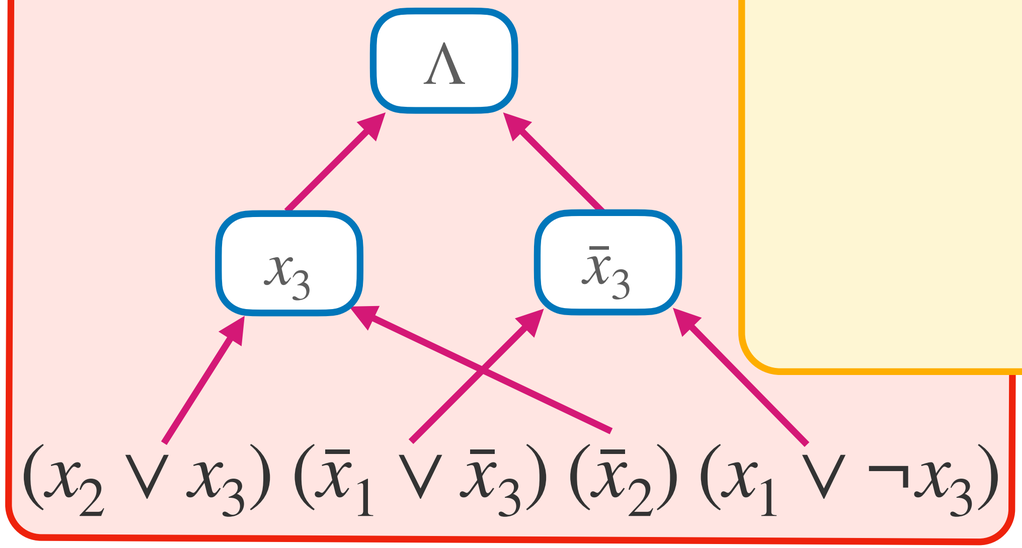

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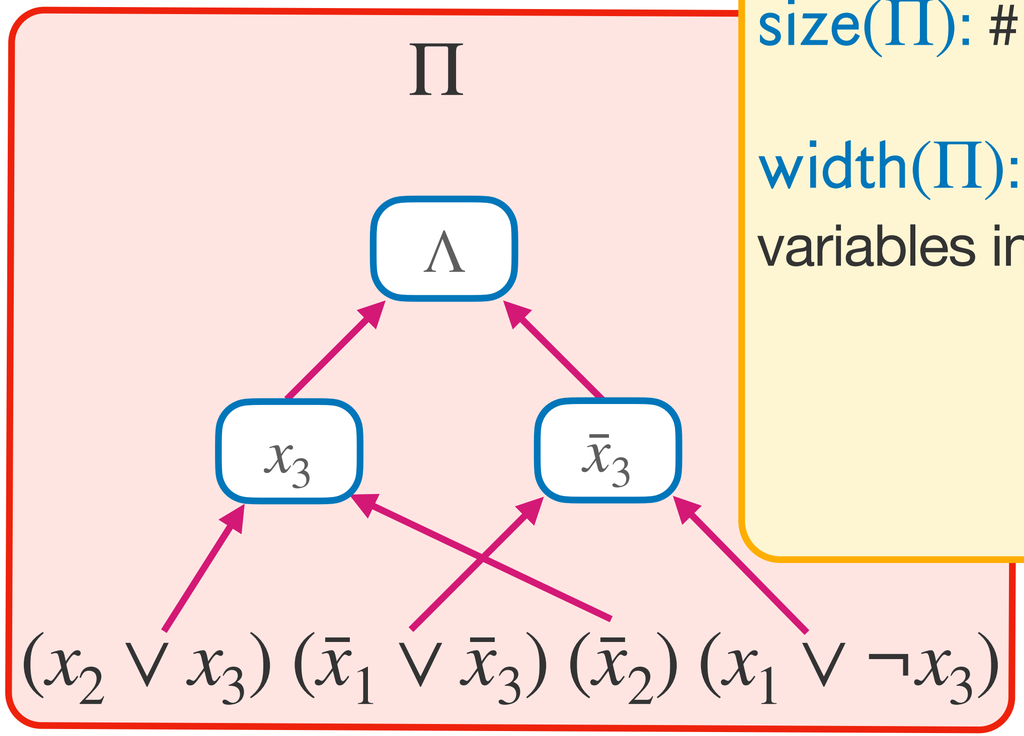

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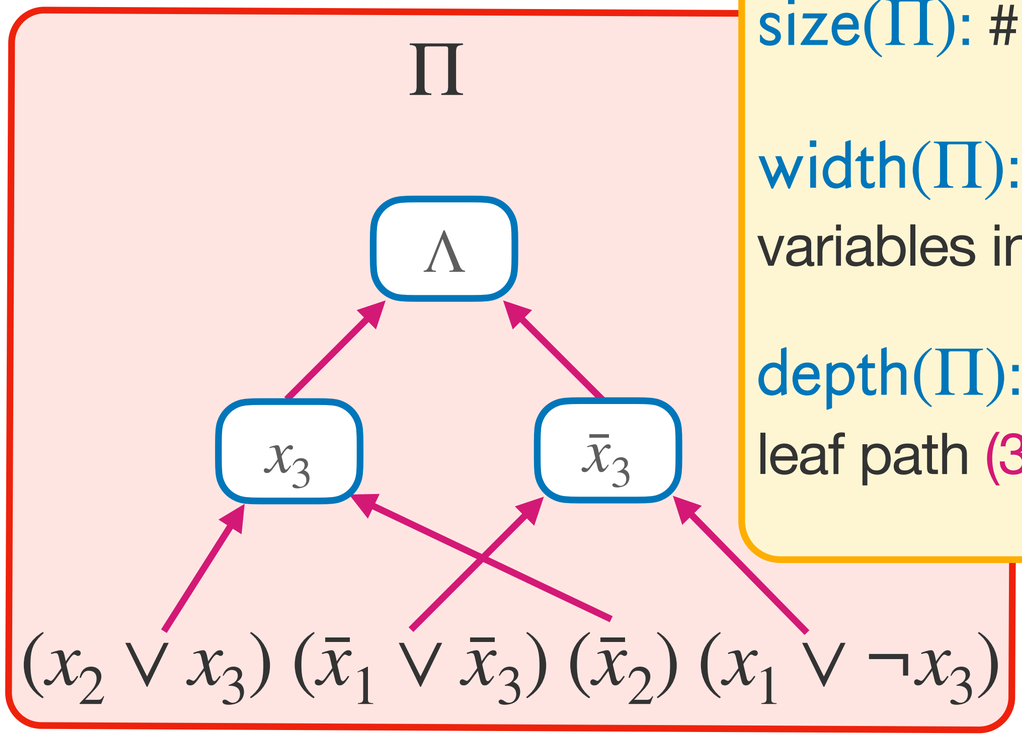

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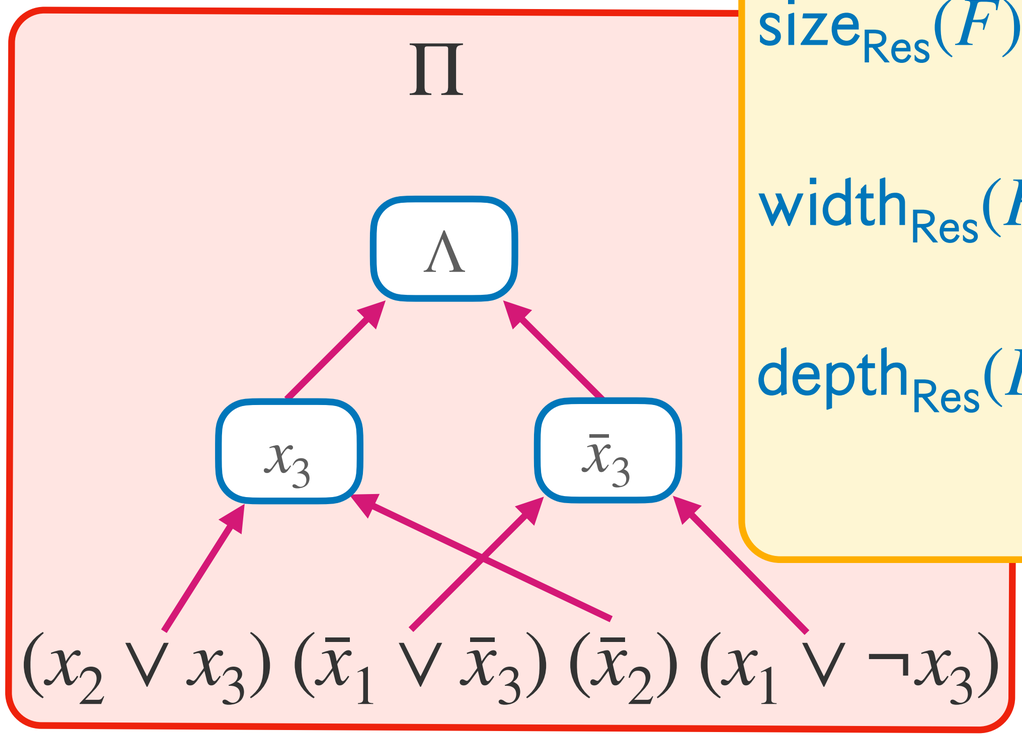

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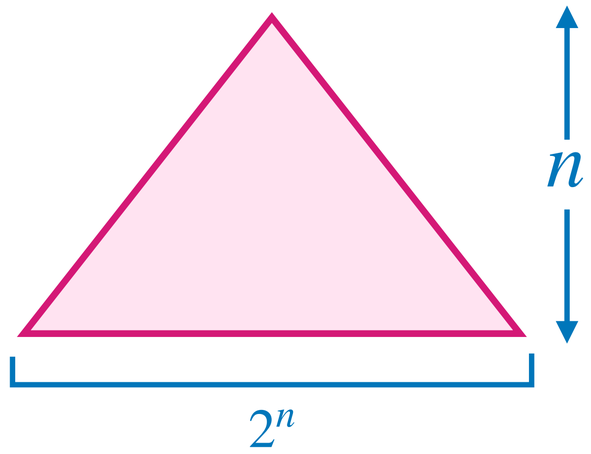

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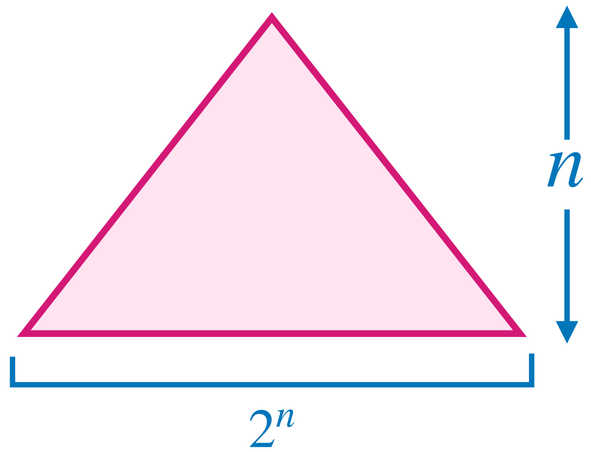

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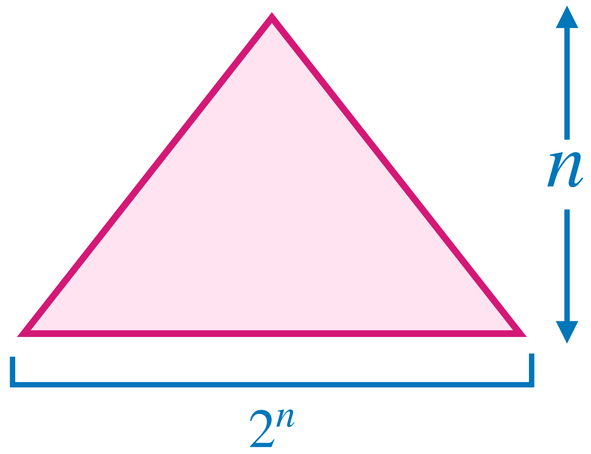

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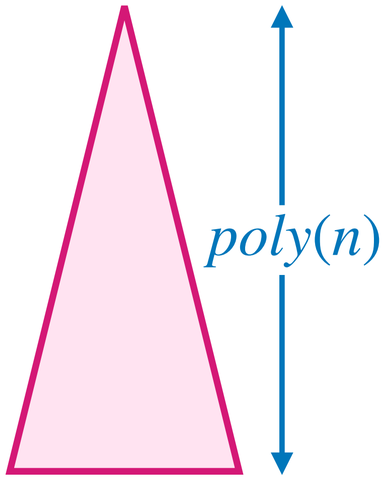
→ Resolution (Res(k), Cutting Planes) cannot always be balanced

This Work

For any $P \in \{$Resolution, Res(k), Cutting Planes$\}$

There is a CNF formula F on n variables such that

- There is a polynomial size P-proof of F
- Any subexponential-size P-proof of F must have polynomial depth



This Work

For any $P \in \{$Resolution, Res(k), Cutting Planes$\}$

There is a CNF formula F on n variables such that

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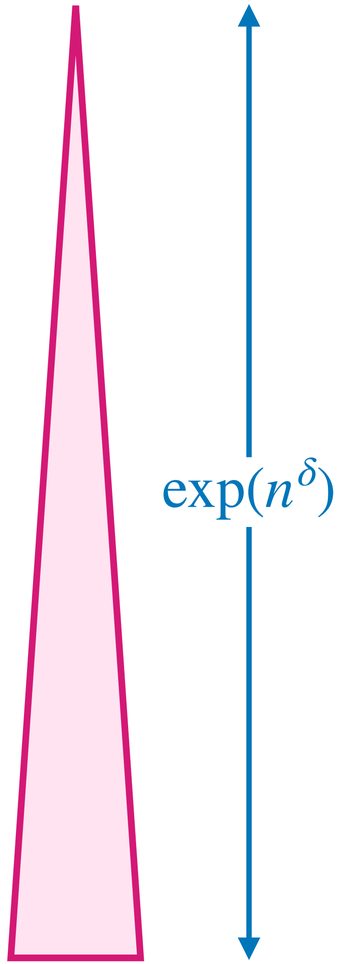

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* Caveat: F has $n^{O(c)}$ many clauses — We'll come back to this later!

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Let P, Q be two proof systems

A lifting theorem relates the complexity of

- P-proofs of F
- Q-proofs of $F \circ g$

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$\rightarrow$ Theme of lifting theorems

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Proof: Let Π be a proof of $F \circ \mathrm{XOR}_2 := F(x_1 \oplus y_1, \ldots, x_N \oplus y_N)$

Let $\rho \in \{0,1,*\}^{2N}$ be generated as follows — Flip a coin for each $i \in [N]$:

- Heads: set x_i to a random bit, set $y_i = *$
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Observe: $F \circ \mathrm{XOR}_2 \restriction \rho = F$ (some variables negated) $\implies \Pi \restriction \rho$ is a proof of F

If C is a clause of width $\geq w := \mathrm{width}_{\mathrm{Res}}(F)$ then $\Pr[C \restriction \rho \neq 1] \leq (3/4)^w$

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Contradiction!

Lifting (Composition)

Typically

- P is a “weak” proof system
- Q is a “strong” proof system

A lifting theorem shows that the most efficient Q-proof of $F \circ g$ is to simulate the most efficient P-proof of F (with some extra overhead to handle g)

Our Lifting

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Proof Idea:

Find a gadget g such that

1. The number of variables n of $F \circ g$ will be much smaller than N
2. Any small-size Resolution proof of $F \circ g$ will require the same depth as proving F

The Gadget

Our gadget will be the XOR function

$F(\mathrm{XOR}(x_1), \ldots, \mathrm{XOR}(x_N))$

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→ Composing will reduce the total number of variables to $n \ll N$

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Let G be an $N \times n$ bipartite graph

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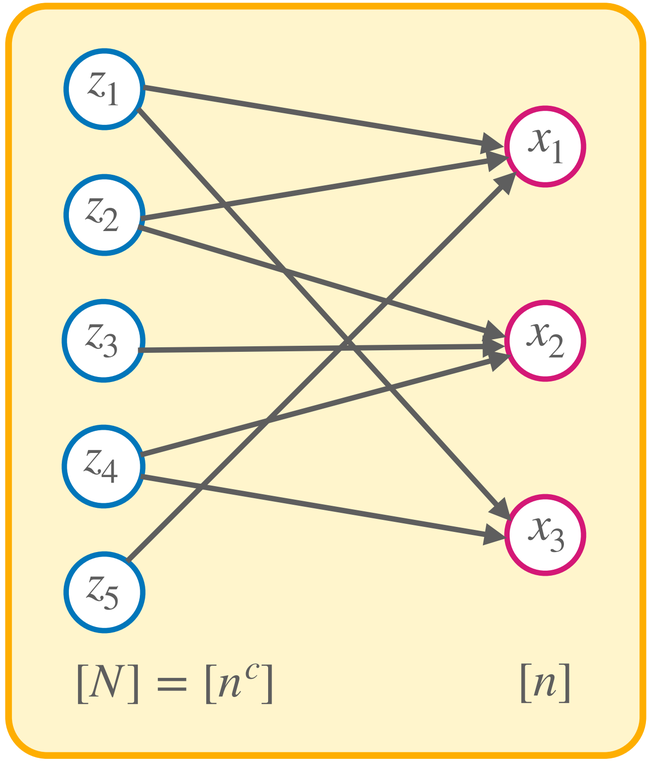

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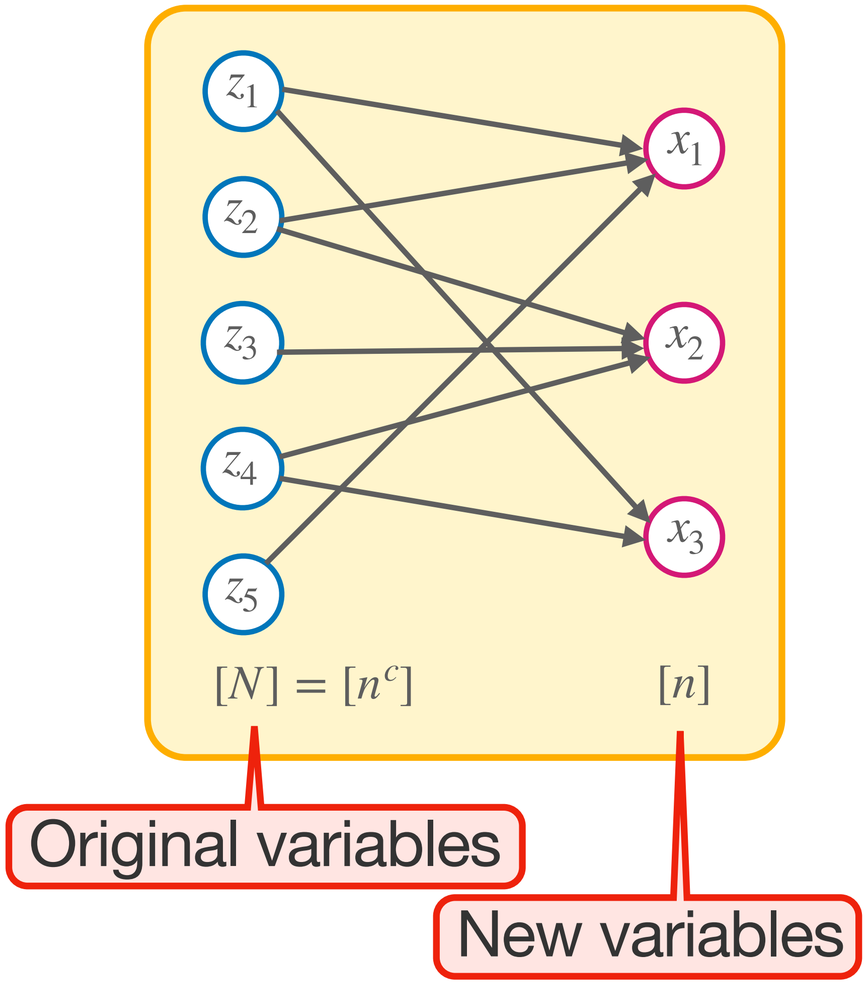

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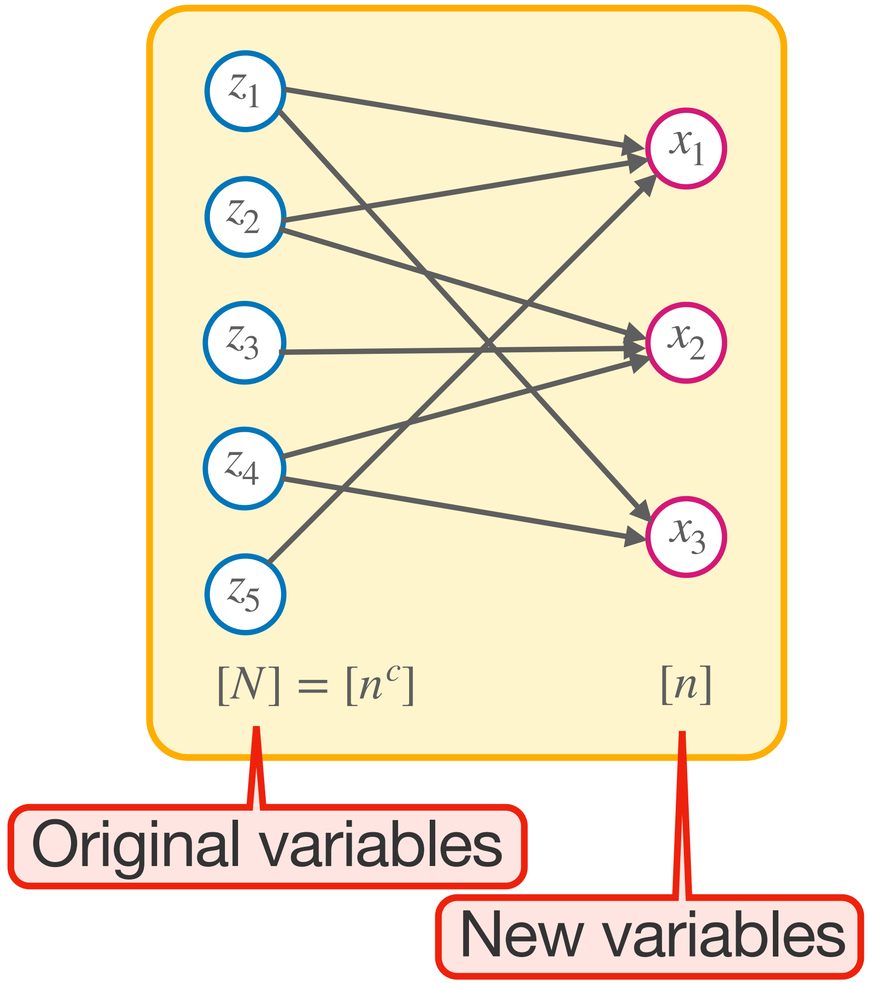

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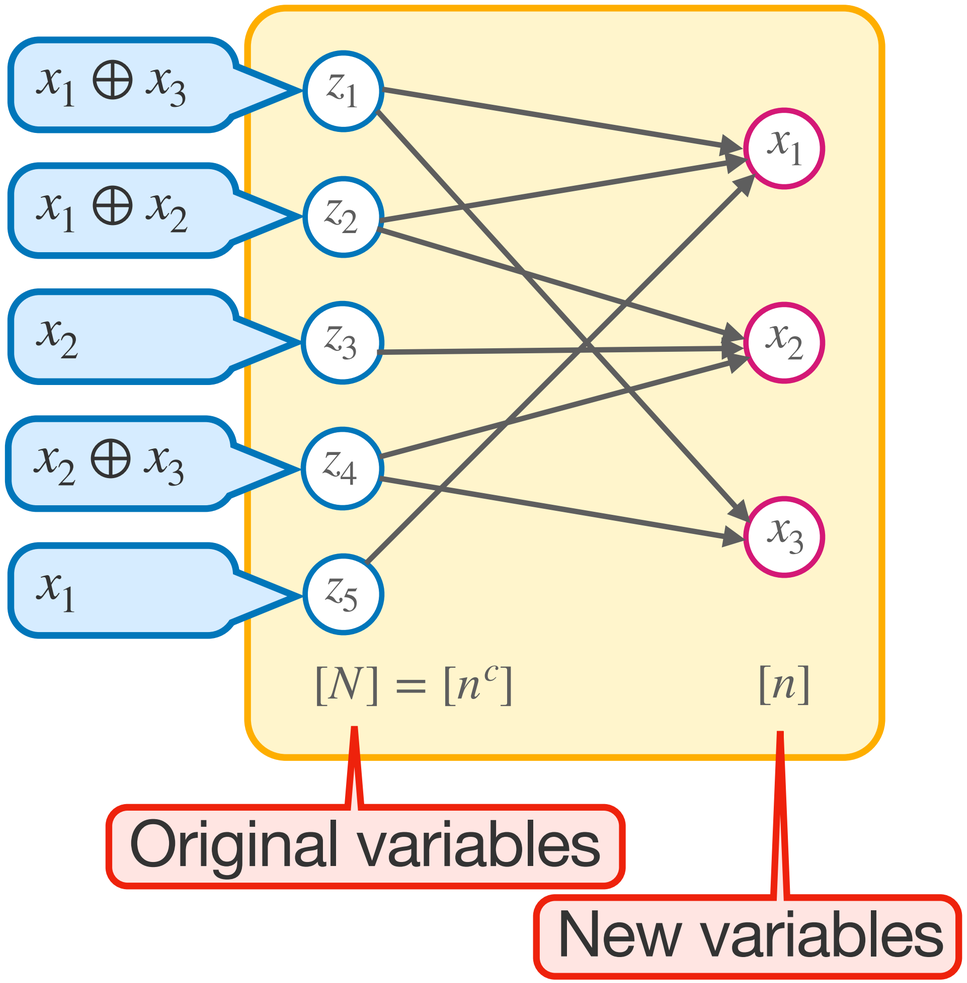

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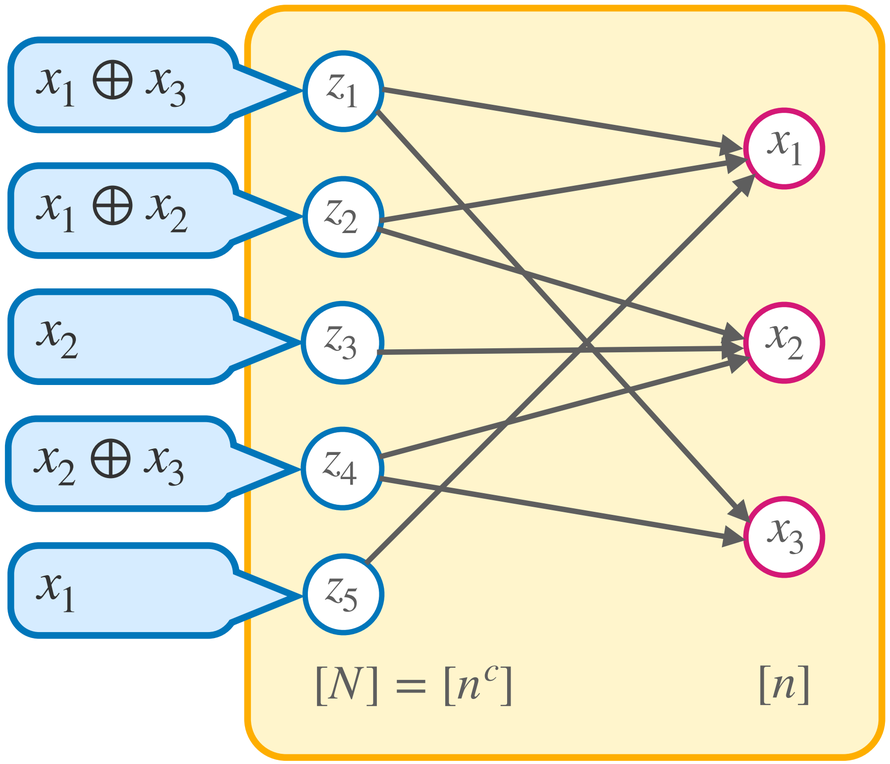

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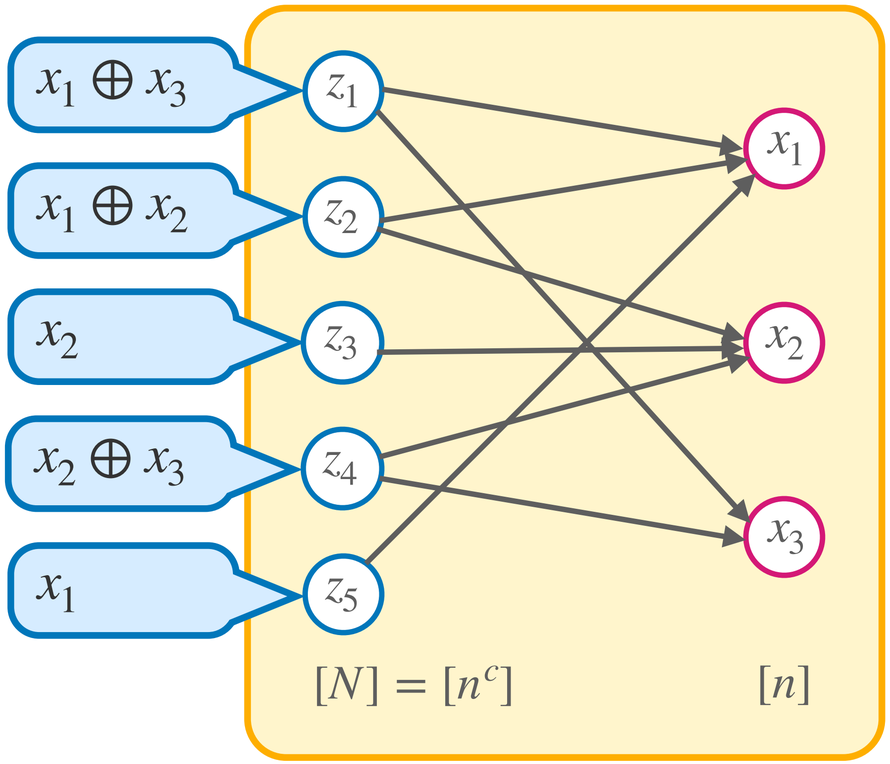

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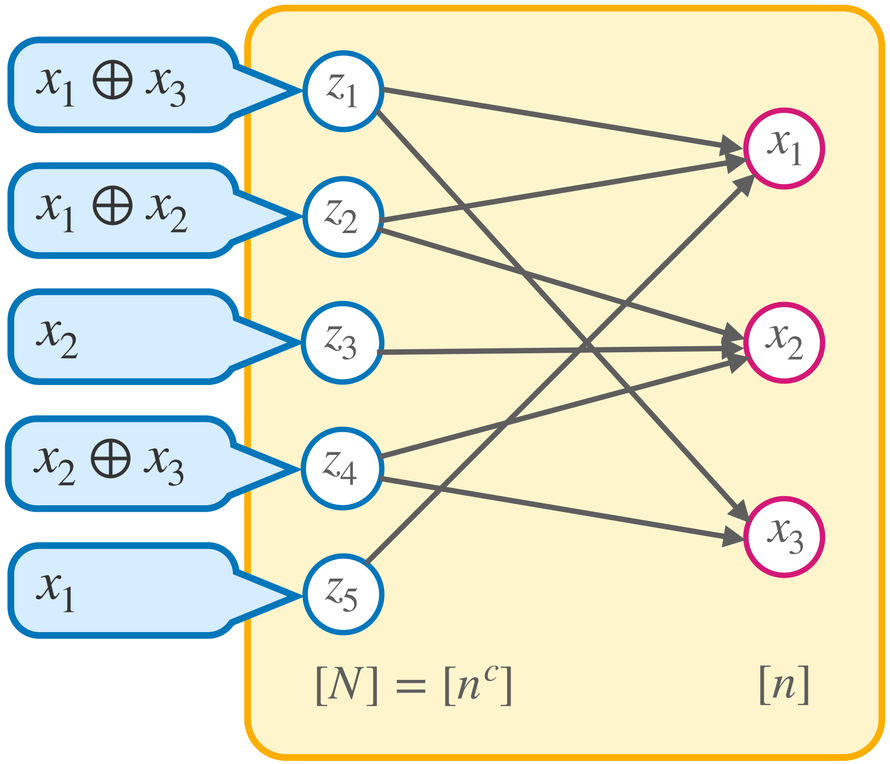

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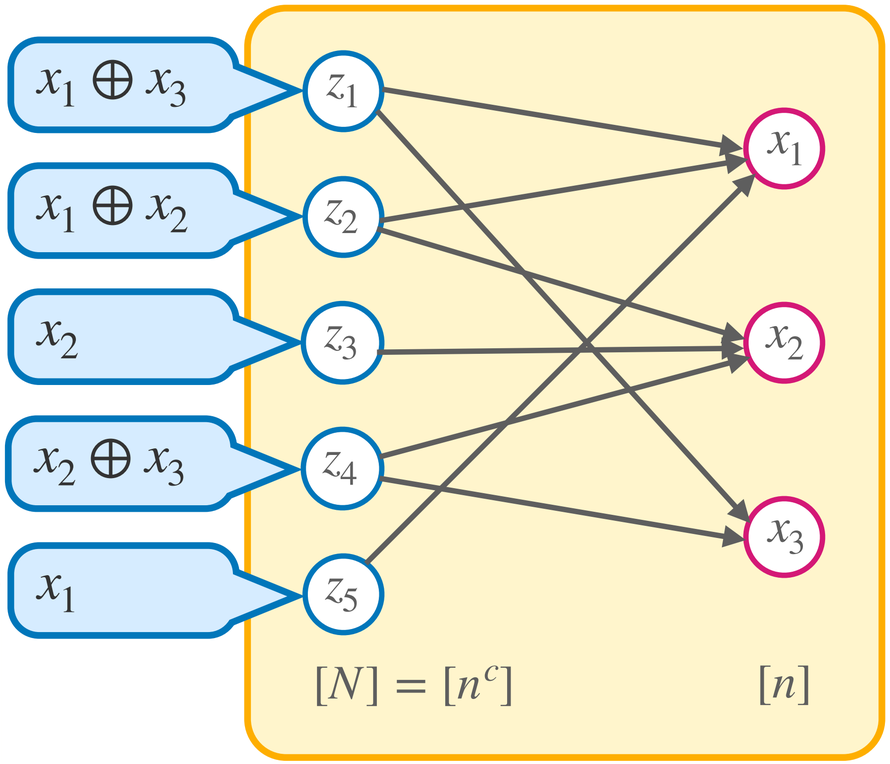

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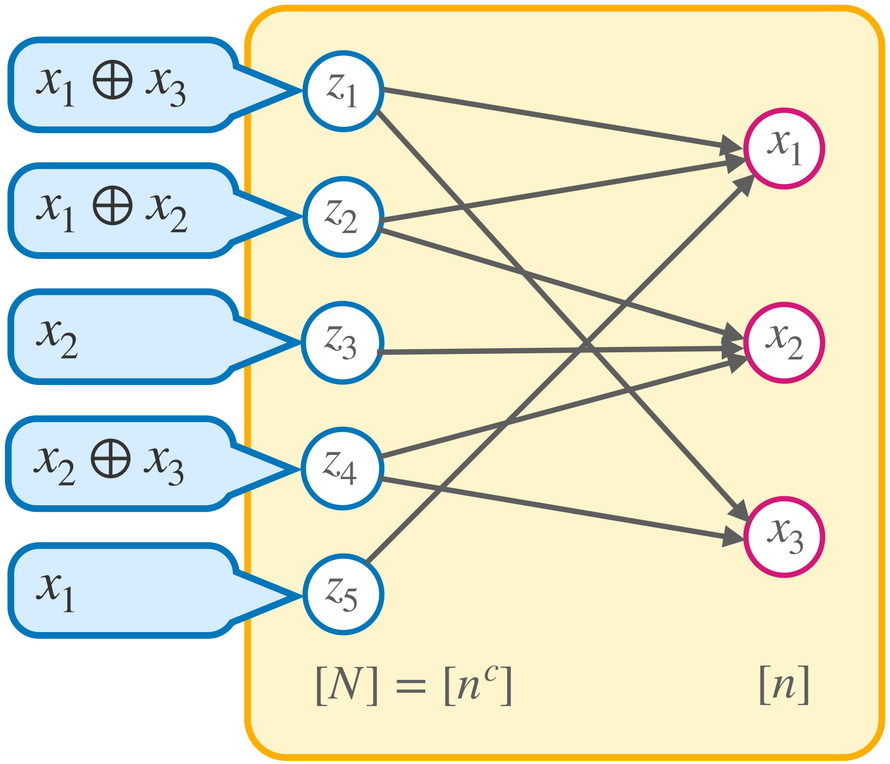

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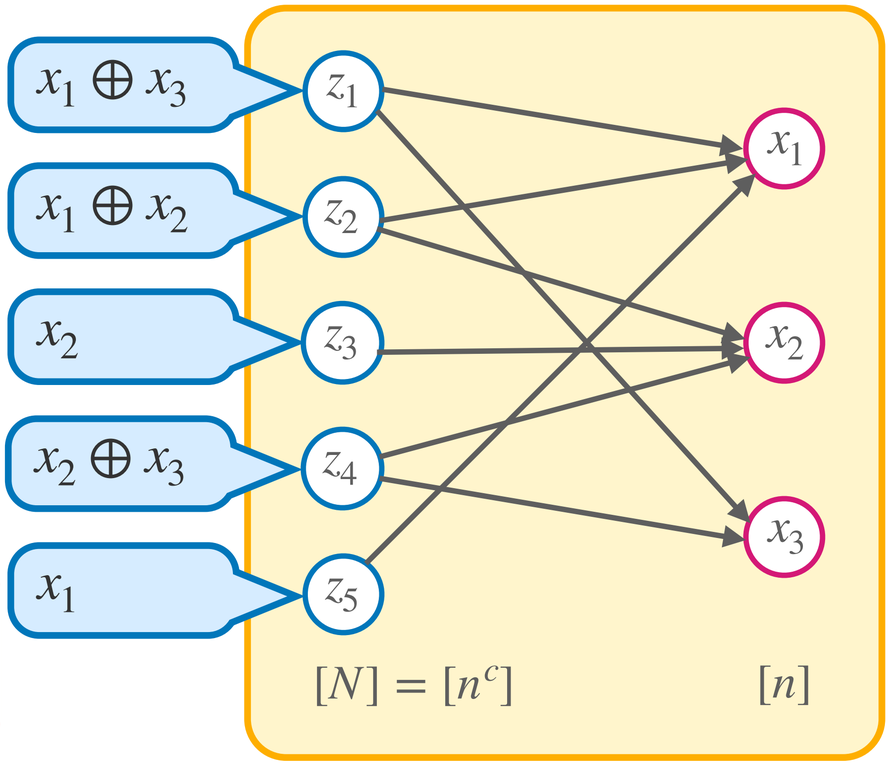

Idea: If the edges of G are sufficiently "spread out"

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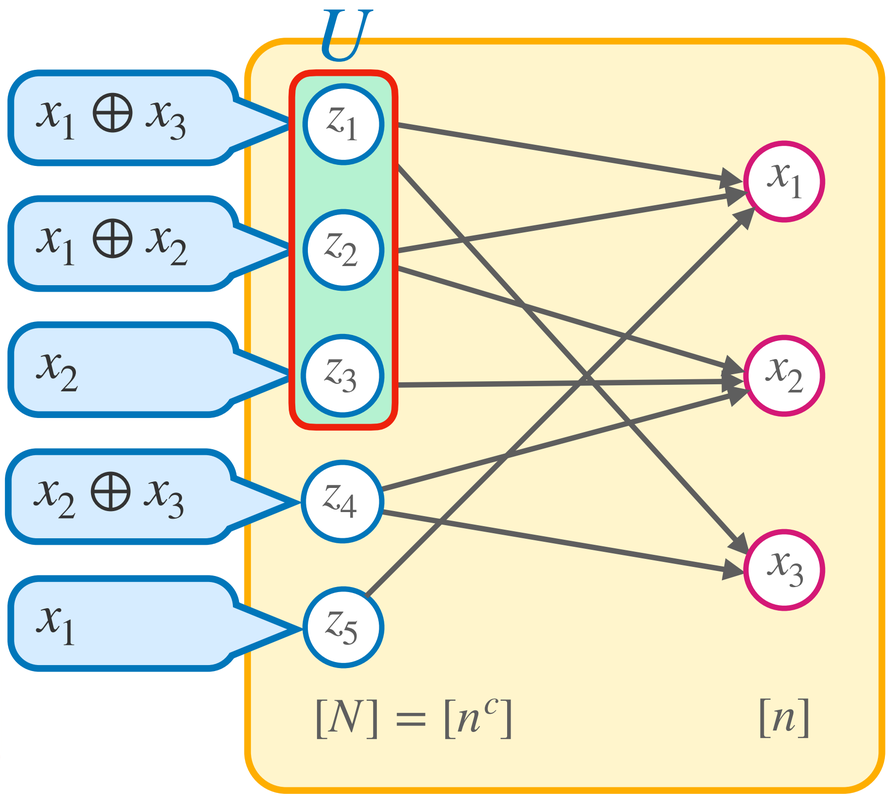

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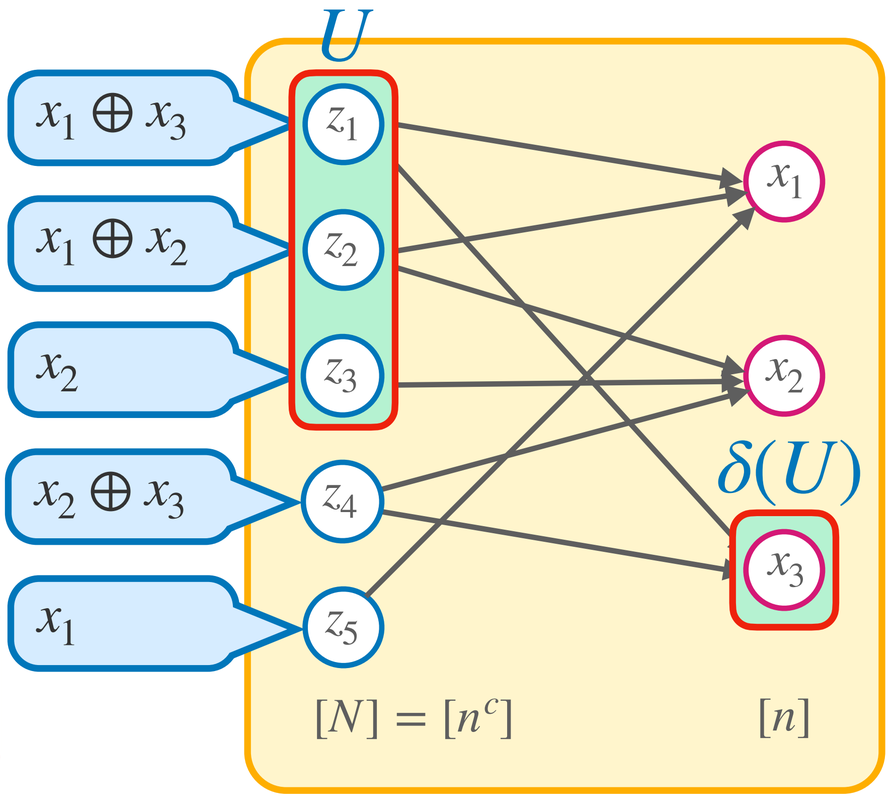

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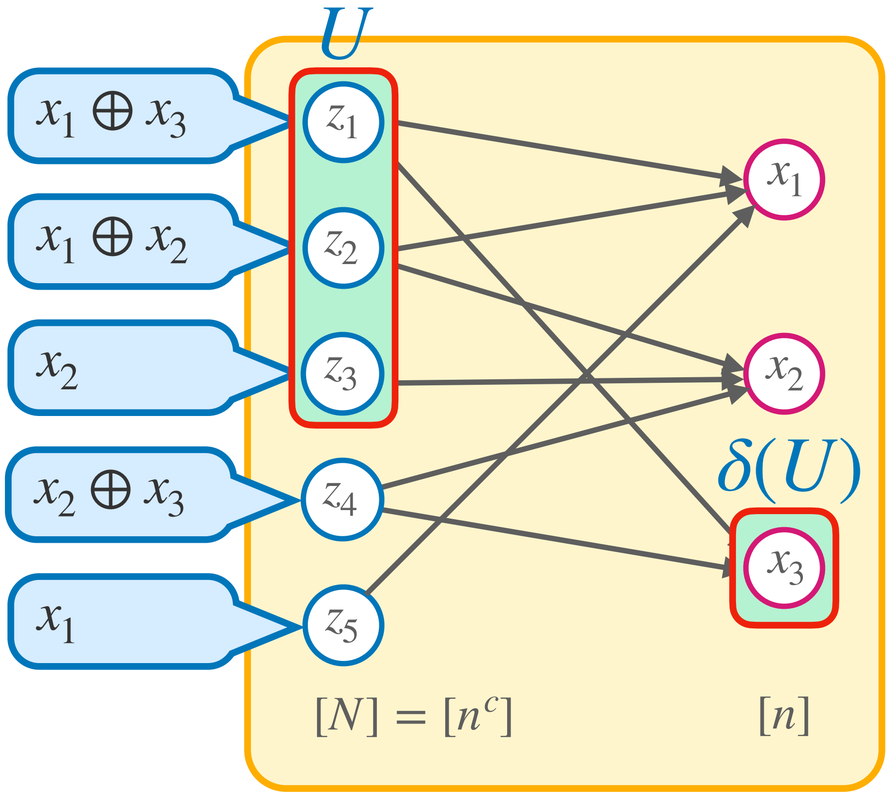

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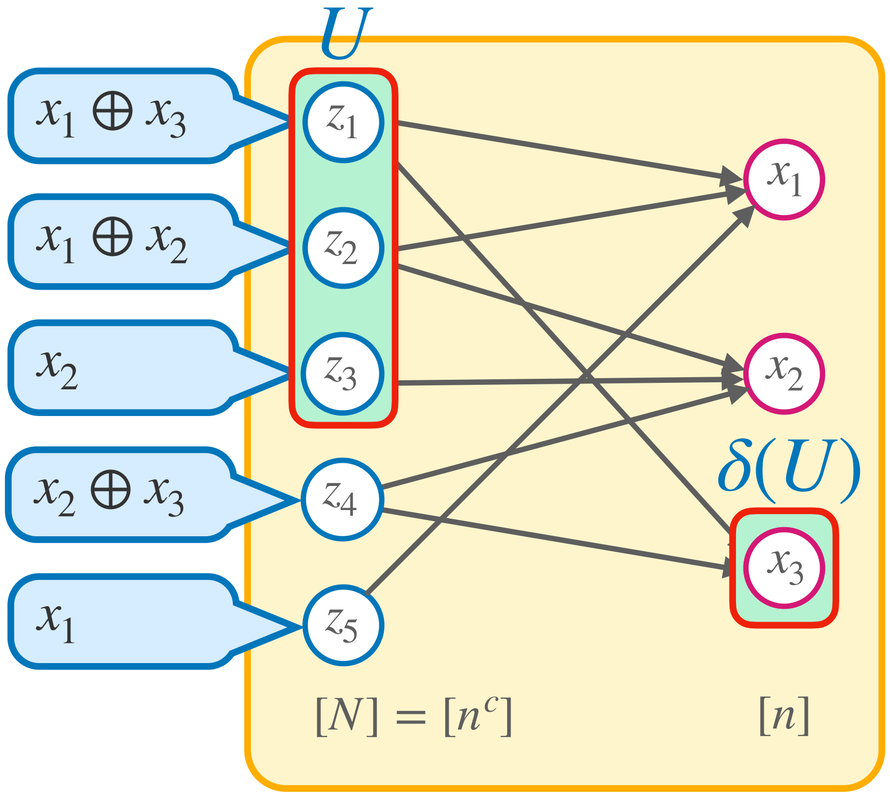

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$\rightarrow$ Our gadget g will be XOR_G for expanding G

Depth Condensation

Main workhorse behind our tradeoff:

Depth Condensation Theorem: ([Razborov16] stated for tree-Res)

Let G be an $(r,2)$-boundary expander, F any unsatisfiable formula.

If Π is a Resolution proof of $F \circ \mathrm{XOR}_G$ with $\mathrm{width}(\Pi) \leq r/4$ then

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→ We give a simple proof

→ Combine this with the **width-to-size lifting theorem** to prove our main tradeoff!

Main Tradeoff (For Resolution)

Let $\varepsilon > 0$, let $c \geq 1$ be real-valued parameter

Main Theorem: There is a CNF formula F on n variables such that

1. There is a P-proof of F of size $n^c \cdot 2^{O(c)}$
2. If Π is a P-proof of F with $\text{size}(\Pi) \leq \exp(o(n^{1-\varepsilon}/c))$ then

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$\rightarrow$ Compose with XOR_2 to translate the **width** bound to **size**

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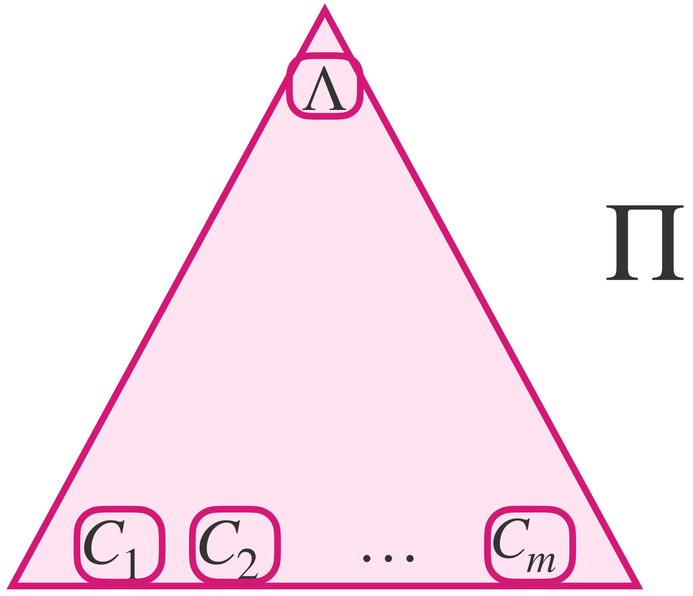

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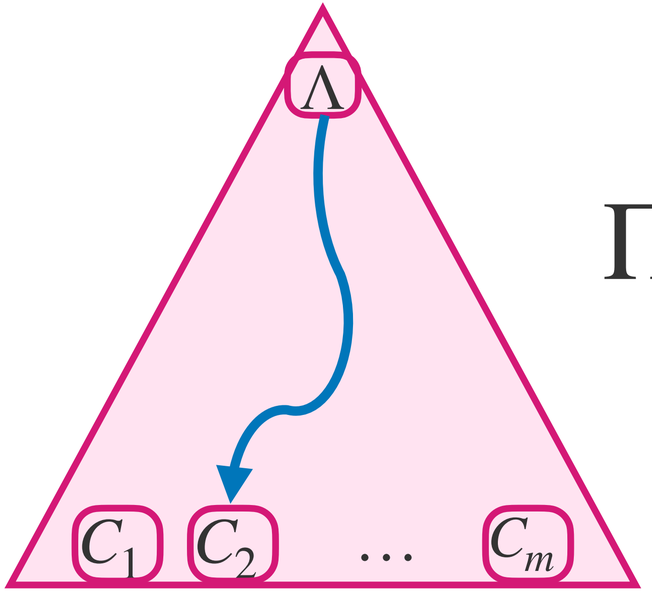

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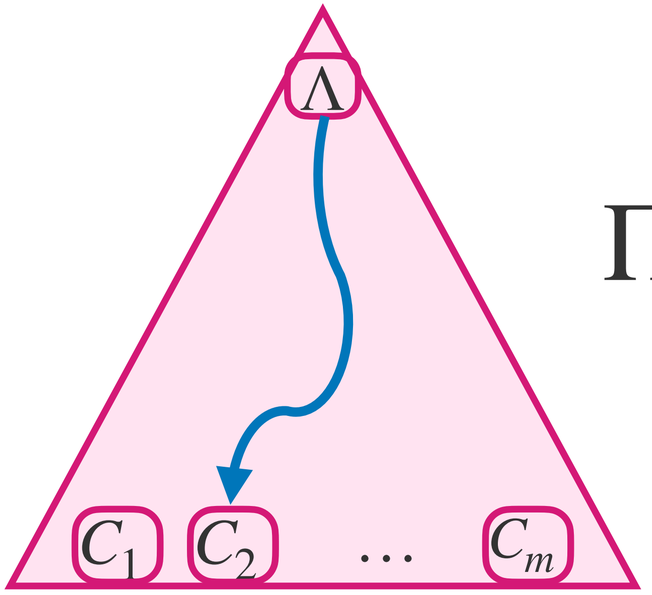

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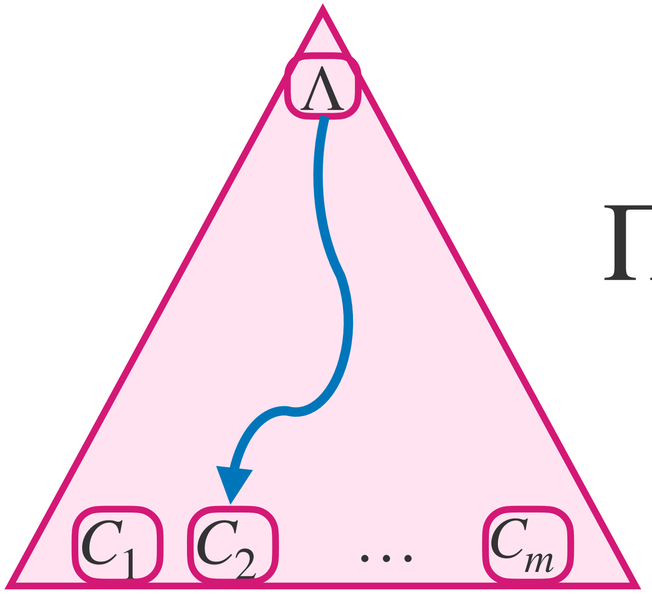

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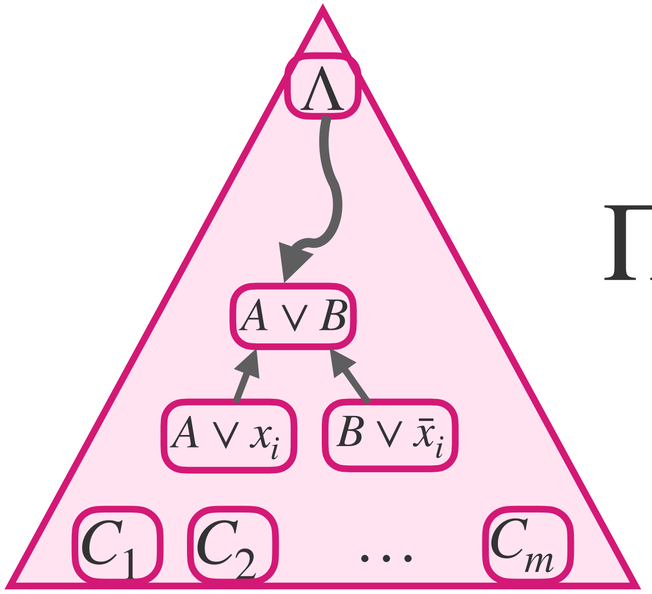

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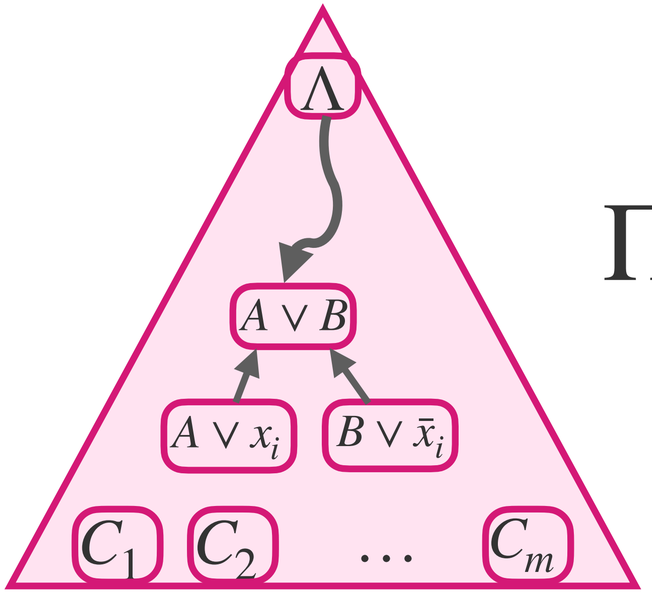

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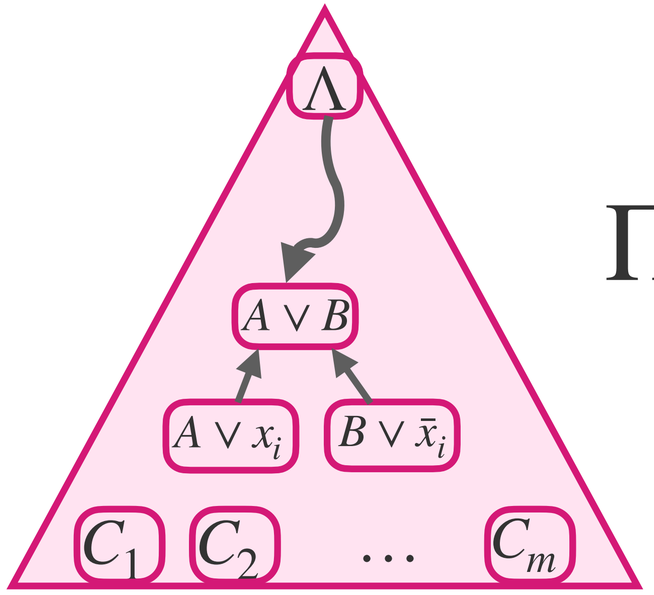

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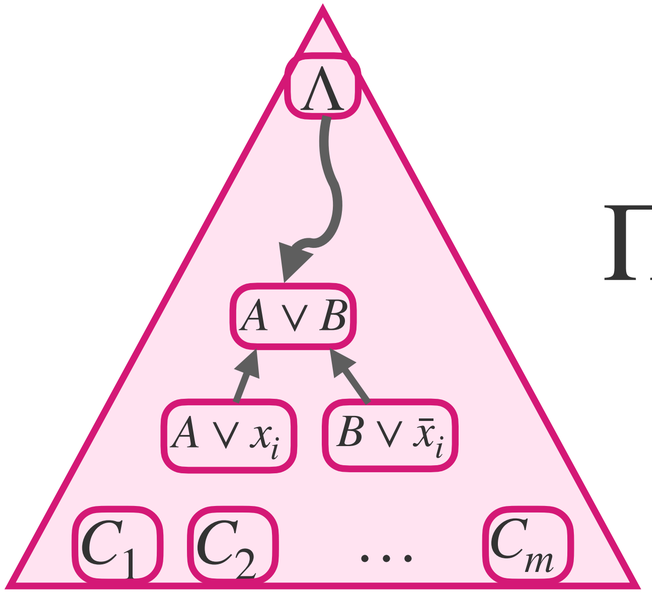

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Depth Condensation Theorem:

Let G be an $(r,2)$-boundary expander, F any unsatisfiable formula.

If Π is a Resolution proof of $F \circ XOR_G$ with $\text{width}(\Pi) \leq r/4$ then

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$\rightarrow$ Use D to construct an Adversary Strategy for the w-bounded game on $F \circ XOR_G$ to survive $\Omega(d/w)$ rounds, for any $w \leq r/4$.

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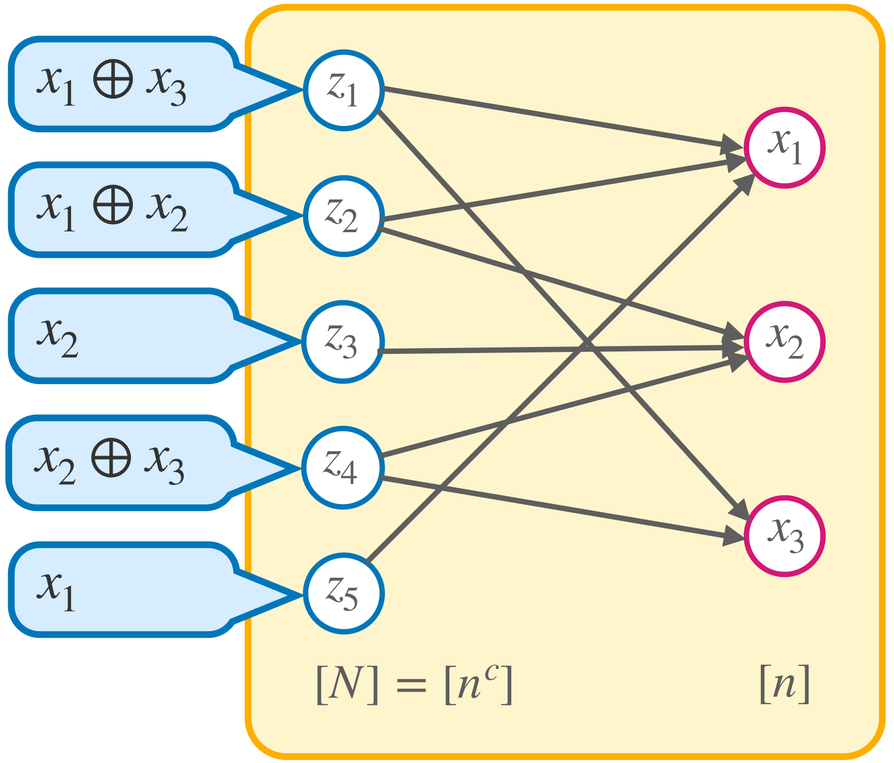

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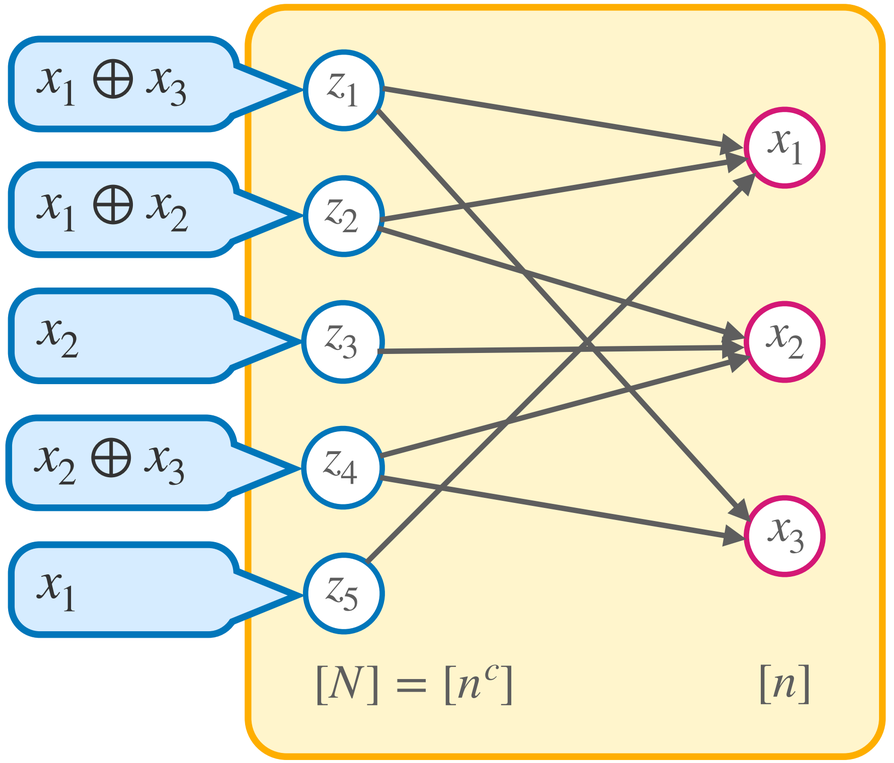

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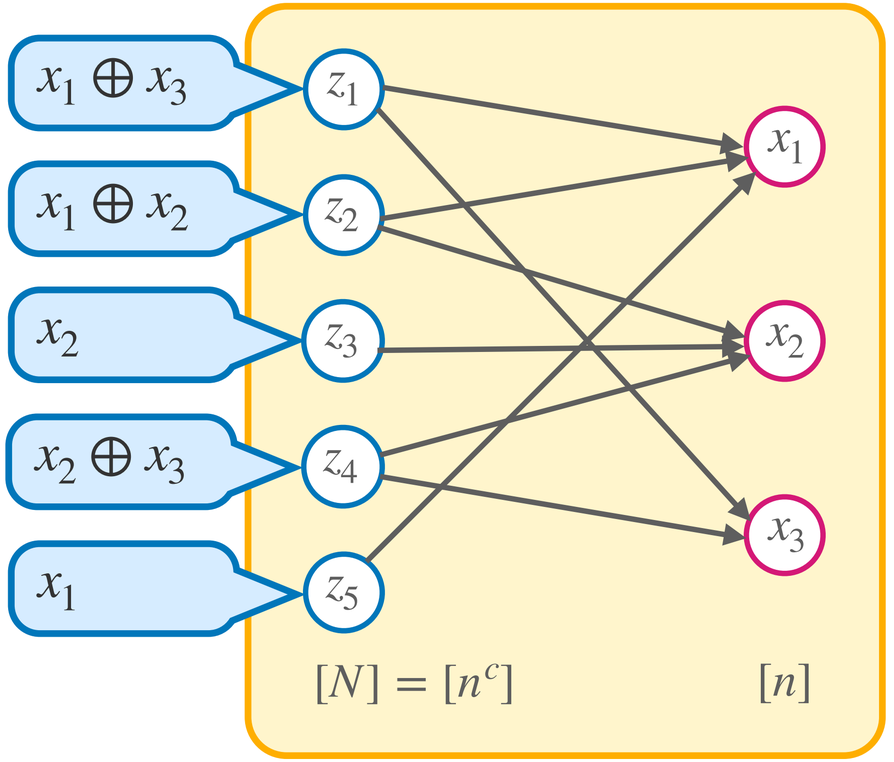

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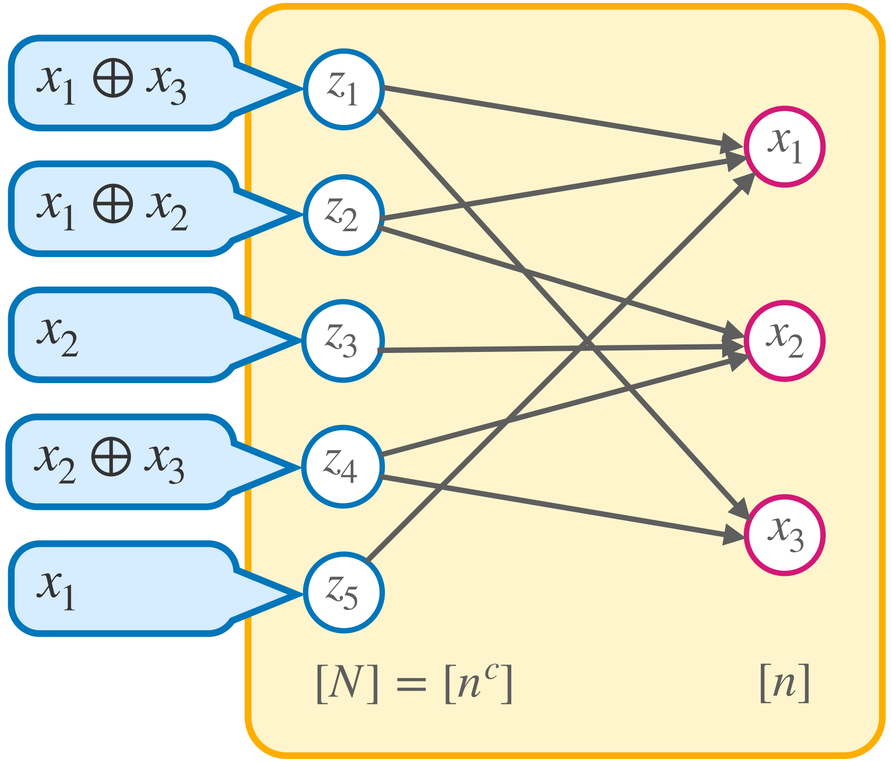

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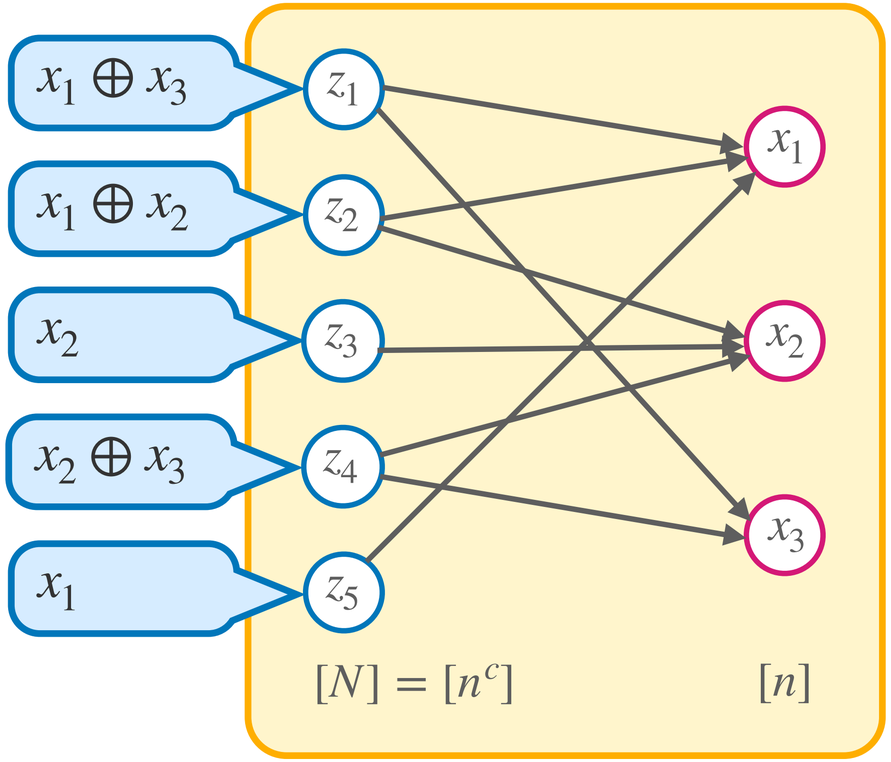

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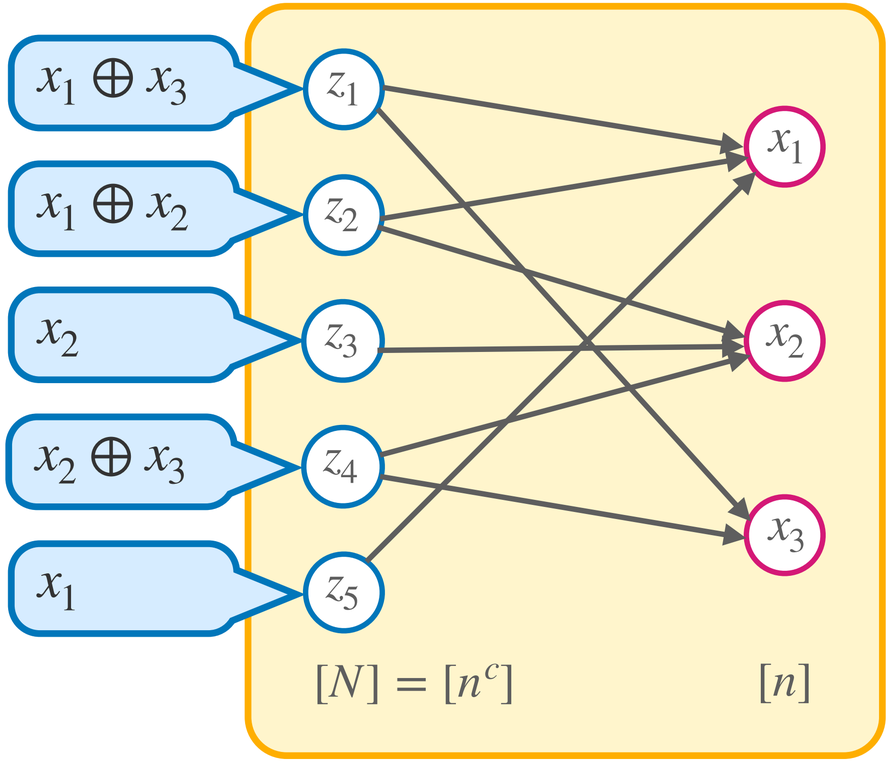

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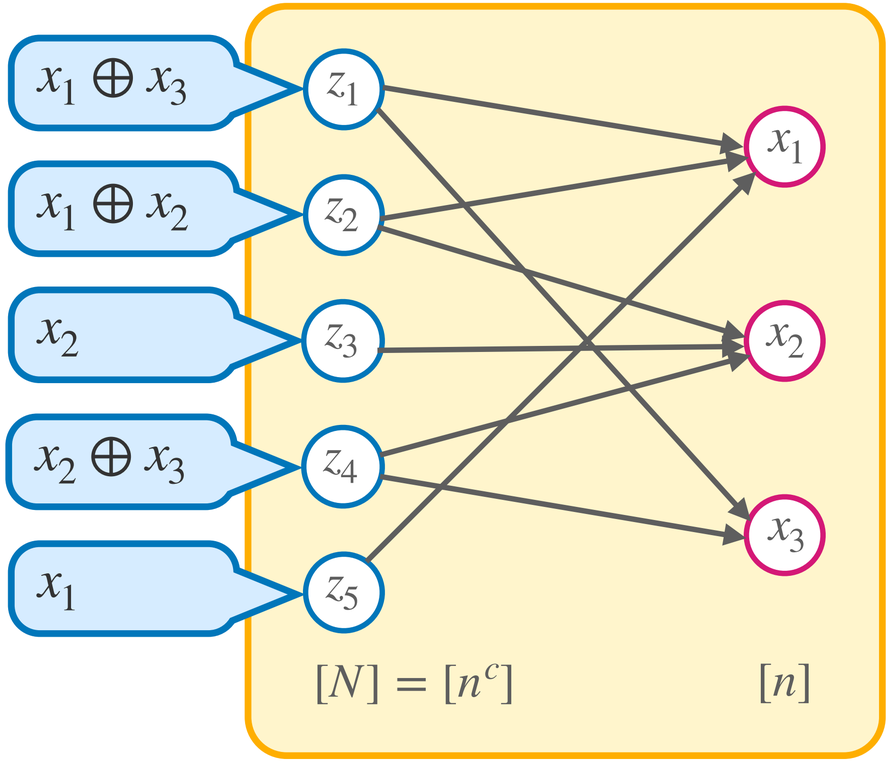

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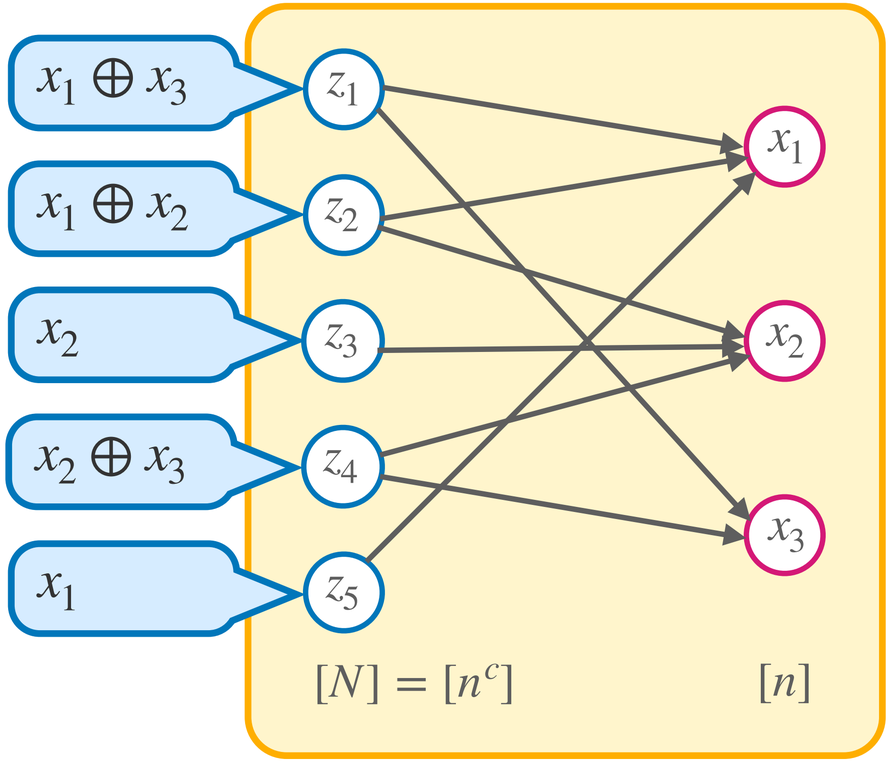

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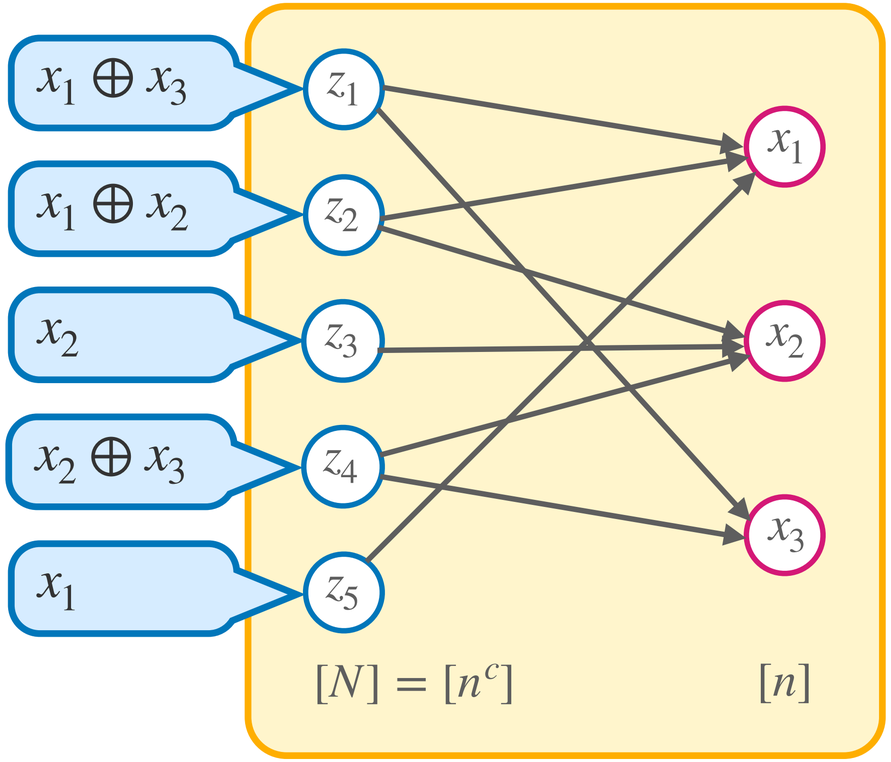
→ Update $\rho = [1,1,*]$ so that $z_2 = x_1 \oplus x_2 = 0$

This forces $z_3 = 1$!

— If on state $[*,0,*,*,1]$ A sets $z_3 = 0$

→ We cannot follow A!



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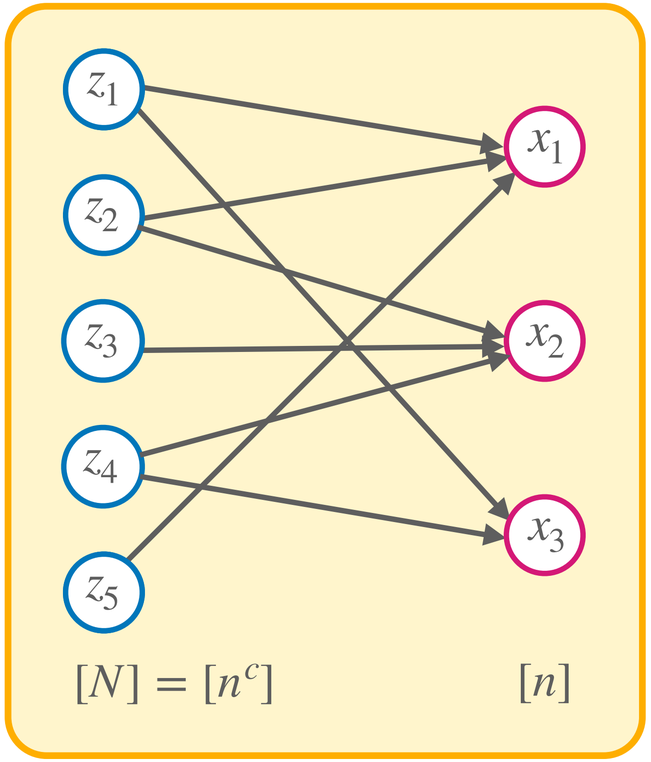

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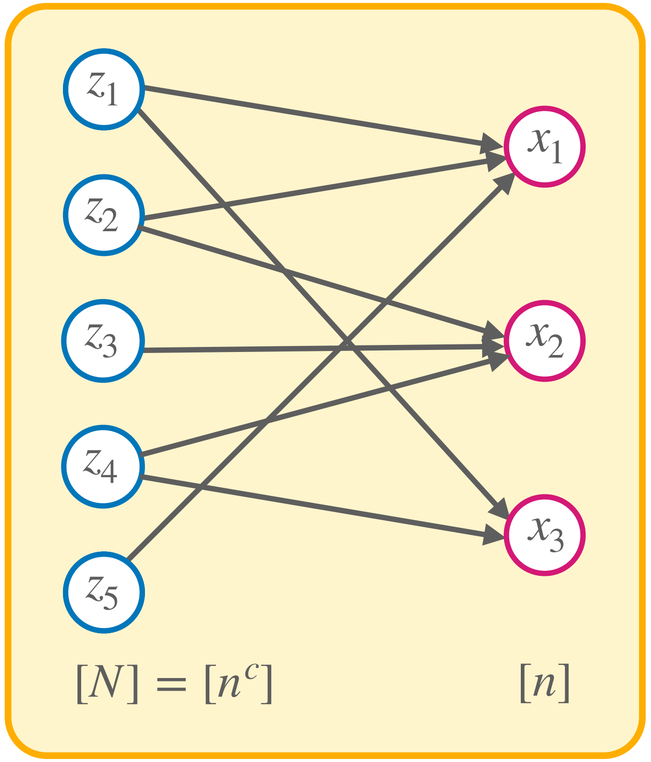

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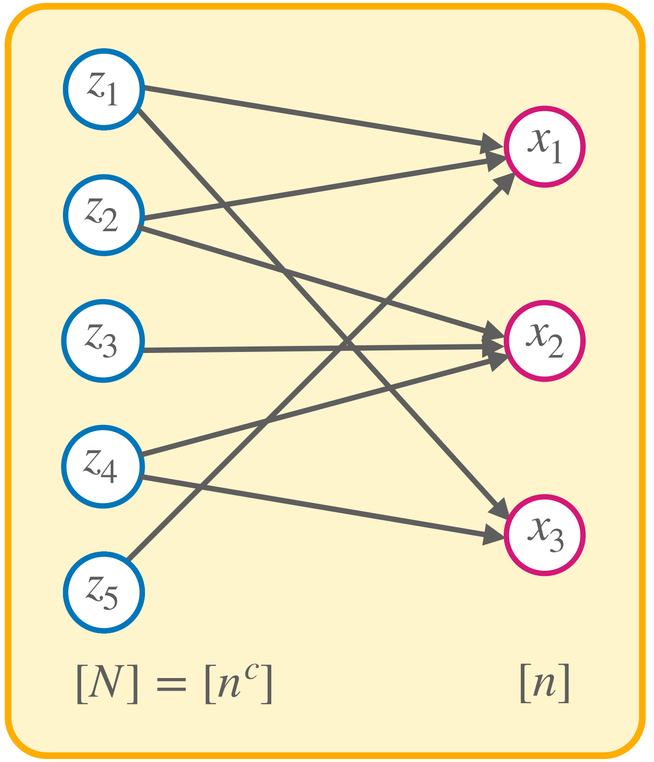

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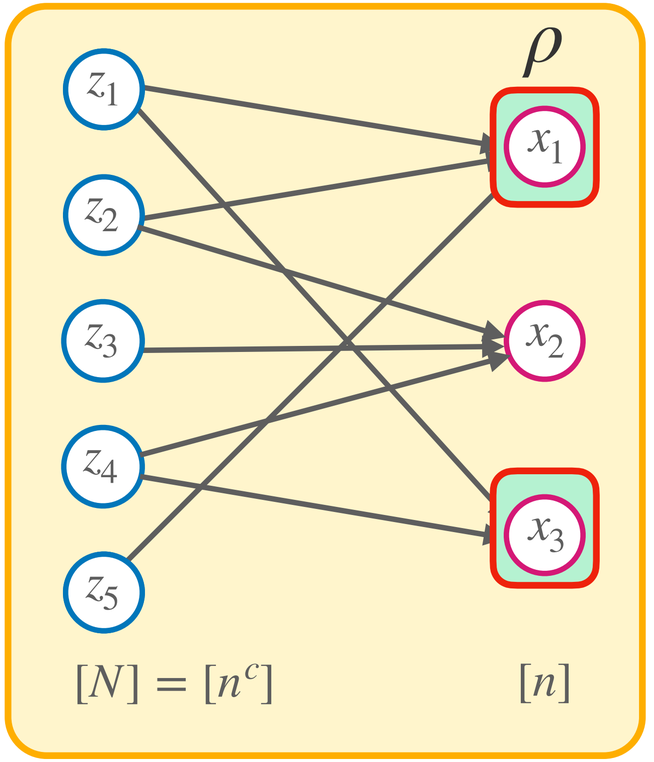

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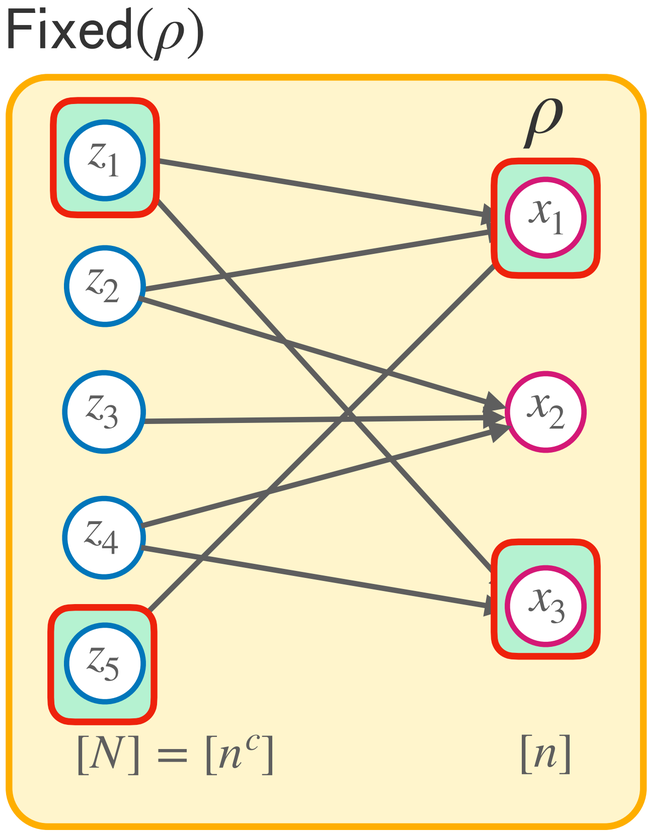

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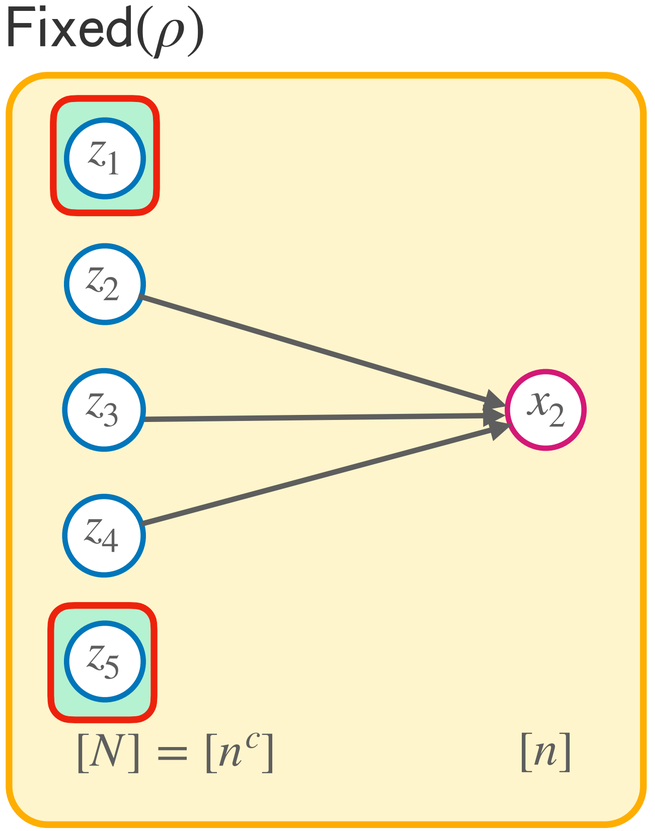

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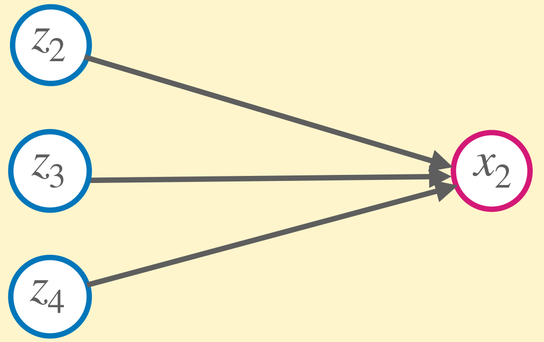

$[N] = [n^c]$ $[n]$

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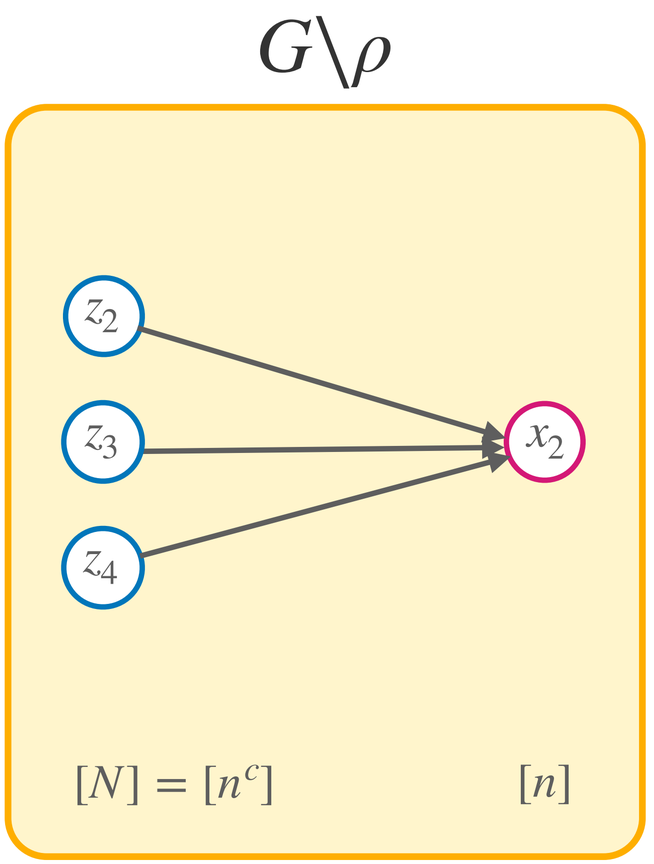

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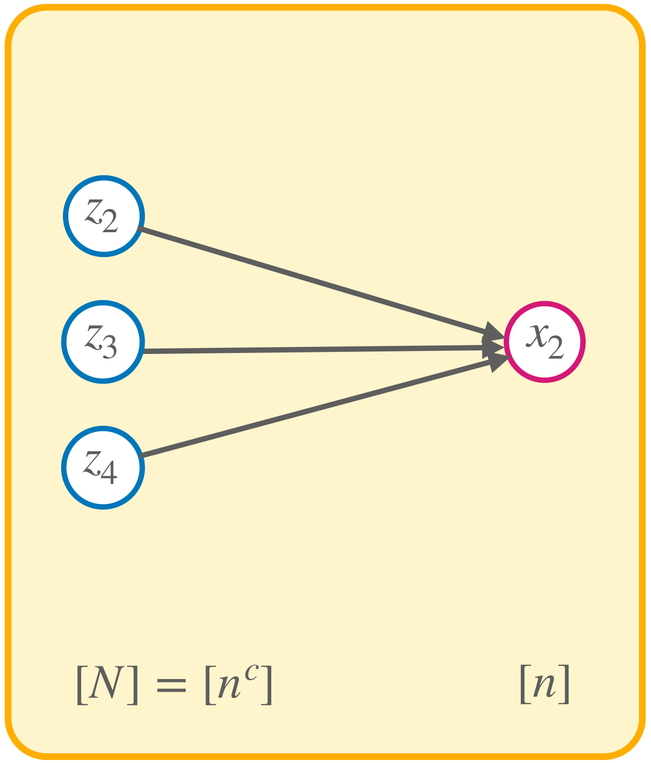

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→ Setting x_i doesn't determine any z-variable

Expansion Restoration

However, after setting x_i, $G \backslash \rho$ may no longer be $(r/2, 3/2)$-expanding…

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$|\mathsf{Cl}(\rho)|$ may be larger than $w = r/4$, but don't worry about that for now

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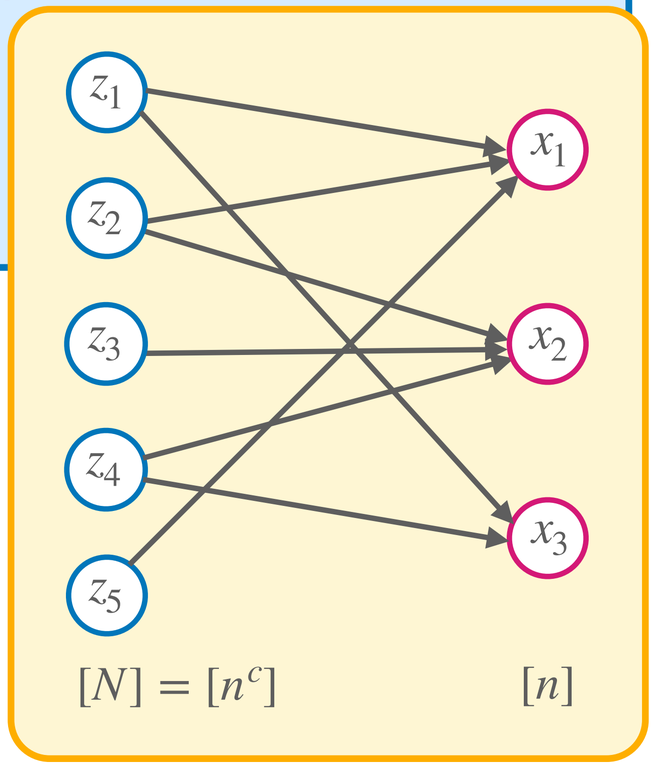

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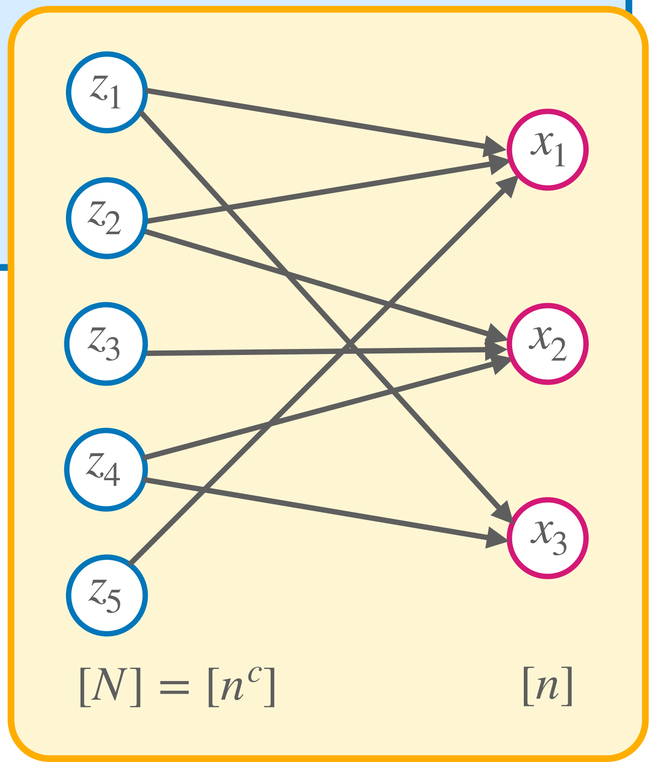

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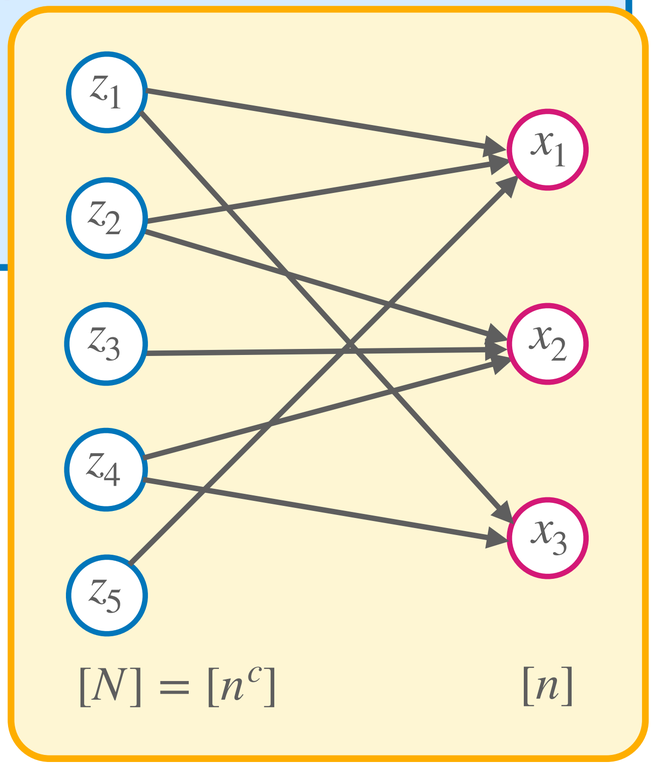

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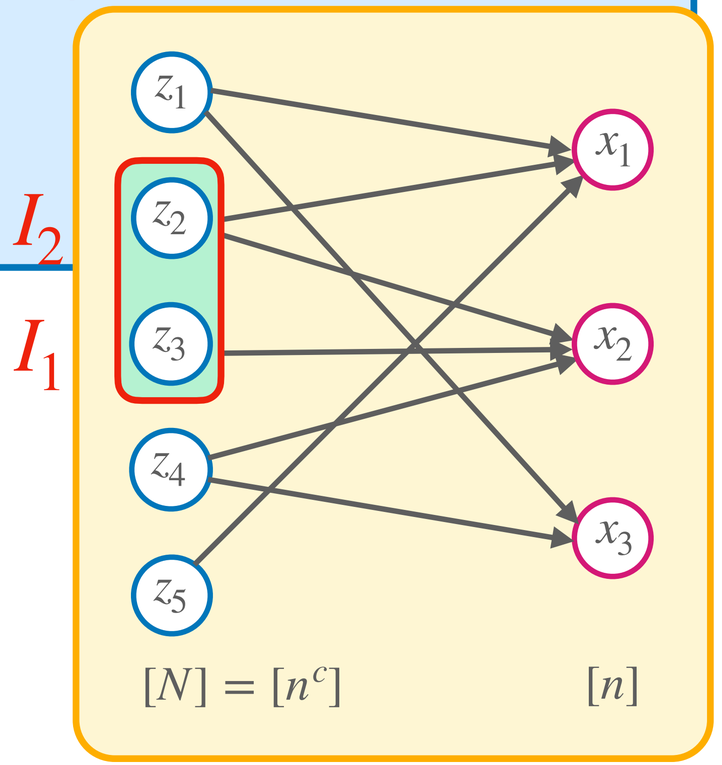

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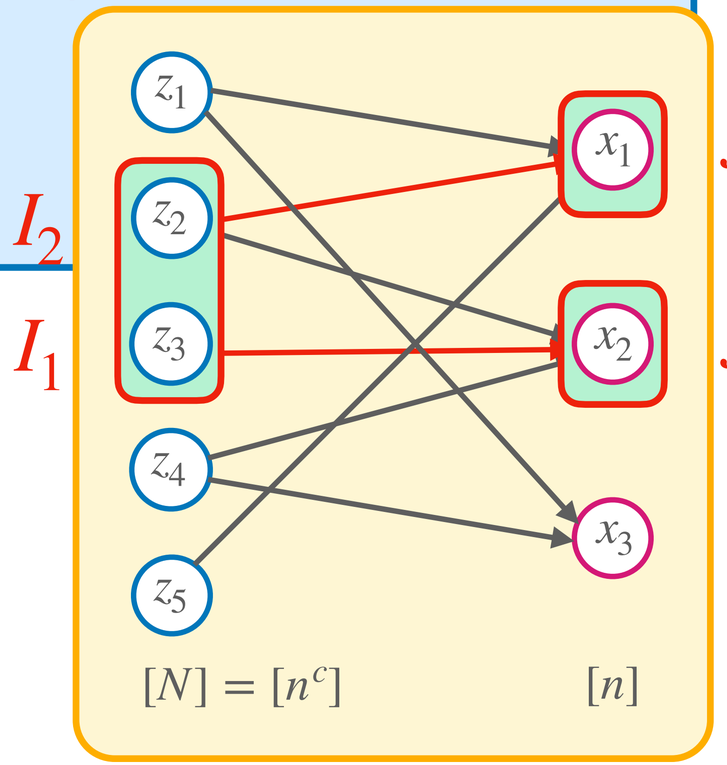

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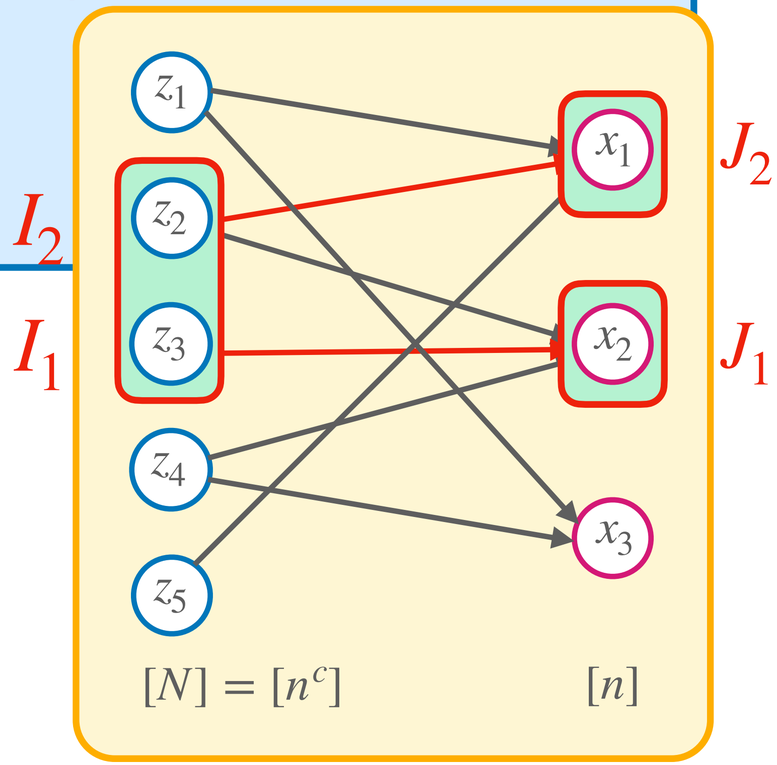

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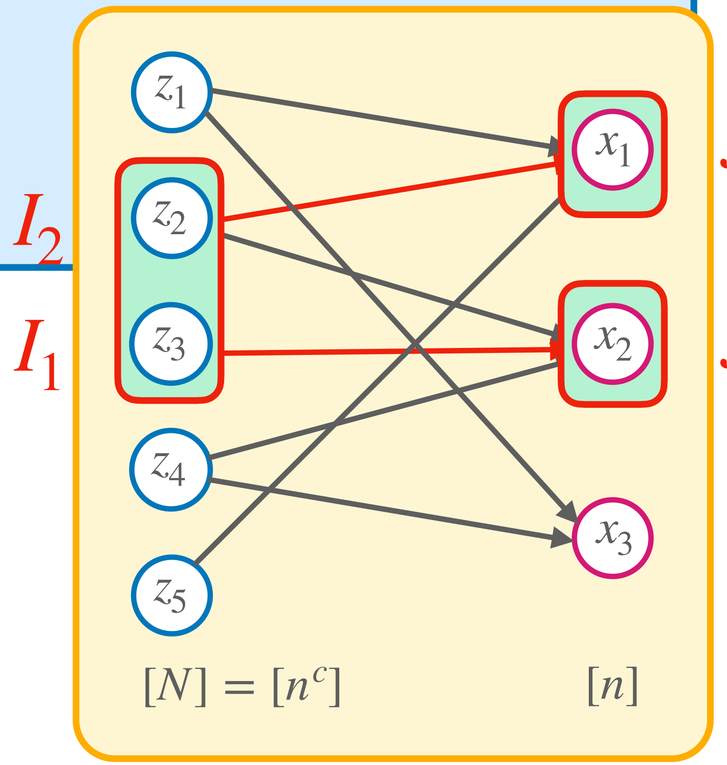

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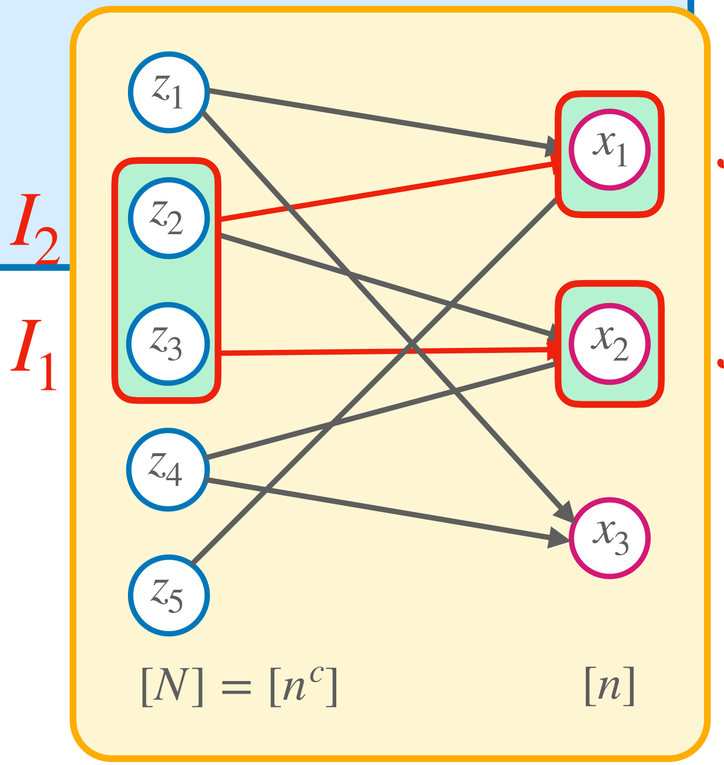

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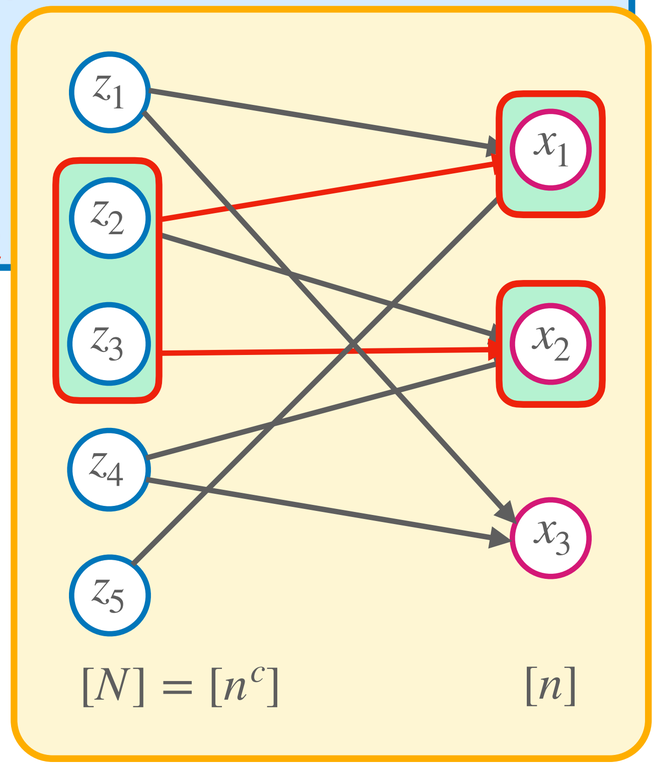

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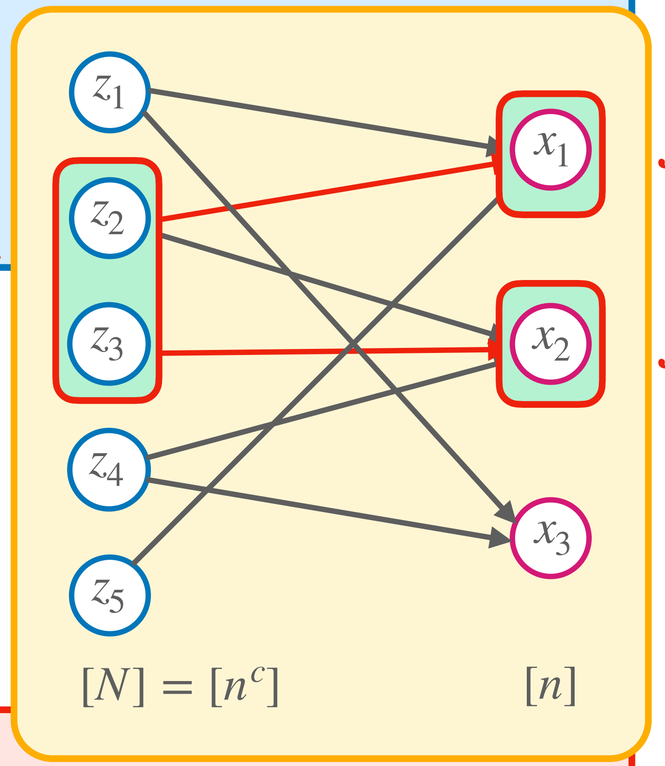

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SDR Lemma:

If $G \backslash \rho$ is a $(r/2, 1/2)$-boundary expander $\Longrightarrow$ any $|I| \leq r/2$ has a strong SDR

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To restore expansion, set the variables in $\mathsf{Cl}(\rho)$ as follows: let A be the adversary for F

RestoreExpansion($\rho, \mathsf{Cl}(\rho)$): Such that $G \backslash \rho$ is a $(r/2, 1/2)$-expander

SDR Lemma: $\mathsf{Fixed}(\mathsf{Cl}(\rho)) \backslash \mathsf{Fixed}(\rho) = \{I_1, \ldots, I_t\}$ has a strong SDR $J = J_1, \ldots, J_t$

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→ By the closure lemma, $G \setminus \rho$ is now a $(r/2, 3/2)$-expander — Invariant restored!

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$\implies$ We query A at most $r/2 = O(w)$ times

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Depth Condensation Theorem

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Let G be an $(r,2)$-boundary expander, F any unsatisfiable formula.

If Π is a Resolution proof of $F \circ XOR_G$ with $\text{width}(\Pi) \leq r/4$ then

$$\text{depth}(\Pi)\text{width}(\Pi) = \Omega(\text{depth}_{\text{Res}}(F))$$

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$\rightarrow$ If no, then supercritical size/depth tradeoffs for monotone circuits follow from the lifting theorem of [GGKS18].